ON ULAM’S TYPE STABILITY

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Quite often (e.g., in applications) we have to do with functions that satisfy some equations only approximately. There arises a natural question what errors we commit when we replace such functions by the exact solutions to those equations. Some tools to evaluate them are provided within the theory of the Ulam (also Hyers-Ulam) type stability.

The issue of Ulam’s type stability of (first, functional, but next also difference, differential and integral) equations has been a very popular subject of investigations for the last nearly fifty years (see, e.g., [3, 8, 9, 10]). The main motivation for it was given by S.M. Ulam in 1940. The following definition somehow describes the main ideas of such stability notion for equations in n variables (R_+ stands for the set of nonnegative reals).

Definition 1. Let A be a nonempty set, (X, d) be a metric space, C ⊂ R_+^n be nonempty, T map C into R_+^A, and F_1, F_2 map a nonempty D ⊂ X^A into X^A. We say that the equation

\[ F_1\varphi(x_1, \ldots, x_n) = F_2\varphi(x_1, \ldots, x_n) \]  

is T-stable provided for every ε ∈ C and \( \varphi \in D \) with

\[ d(F_1\varphi_0(x_1, \ldots, x_n), F_2\varphi_0(x_1, \ldots, x_n)) \leq \varepsilon(x_1, \ldots, x_n), \quad x_1, \ldots, x_n \in A, \]

there is a solution \( \varphi \in D \) of equation (1) such that \( d(\varphi(x), \varphi_0(x)) \leq T\varepsilon(x) \) for \( x \in A \).

The next two theorems contain examples of some results on stability of the additive Cauchy equation (see [3]) and of a linear difference equation of higher order (see [7]).

Theorem 1. Let \( E_1 \) and \( E_2 \) be two normed spaces, \( c \geq 0 \) and \( p \neq 1 \) be fixed real numbers. Let \( f : E_1 \to E_2 \) be such that

\[ \|f(x+y) - f(x) - f(y)\| \leq c(\|x\|^p + \|y\|^p), \quad x, y \in E_1 \setminus \{0\}. \]

If \( p < 0 \), then \( f \) is additive (i.e., \( f(x+y) = f(x) + f(y) \) for \( x, y \in E_1 \)). If \( p \geq 0 \) and \( E_2 \) is complete, then there is a unique additive \( T : E_1 \to E_2 \) with

\[ \|f(x) - T(x)\| \leq \frac{c\|x\|^p}{2^{p-1} - 1}, \quad x \in E_1 \setminus \{0\}. \]

Theorem 2. Let \( T \) be either \( \mathbb{N} \) or \( \mathbb{Z} \), \( X \) be a Banach space over \( F = \{\mathbb{R}, \mathbb{C}\} \), \( (b_n)_{n \in T} \) be a sequence in \( X \), \( a_1, \ldots, a_m \in F \), \( \delta > 0 \) and \( r_1, \ldots, r_m \in \mathbb{C} \) be the roots of the characteristic equation of the difference equation

\[ x_{n+m} = a_1x_{n+m-1} + \cdots + a_mx_n + b_n, \quad n \in T. \]  

Suppose that \( |r_i| \neq 1 \) for \( i = 1, \ldots, m \) and \( (y_n)_{n \in T} \) is a sequence in \( X \) with

\[ \|y_{n+m} - a_1y_{n+m-1} - \cdots - a_my_n - b_n\| \leq \delta, \quad n \in T. \]

Then there exists a sequence \( (x_n)_{n \in T} \) in \( X \) such that (2) holds and

\[ \|y_n - x_n\| \leq \frac{\delta}{|1 - |r_1|| \cdots |1 - |r_m||}, \quad n \in T. \]

The lecture contains some basic motivations, definitions and results connected with the notion of the Ulam (but also the Hyers-Ulam) type stability. A general method will also be presented for investigations of that stability, e.g., of the following linear (difference, differential, functional) equations of higher orders:

\[ b_m\varphi(n+m) + b_{m-1}\varphi(n+m-1) + \cdots + b_1\varphi(n+1) + b_0\varphi(n) = G(n), \]
\[ b_m \varphi^{(m)}(z) + b_{m-1} \varphi^{(m-1)}(z) + \cdots + b_1 \varphi'(z) + b_0 \varphi(z) = G(z), \]
\[ b_m \varphi(f^m(z)) + b_{m-1} \varphi(f^{m-1}(z)) + \cdots + b_1 \varphi(f(z)) + b_0 \varphi(z) = G(z). \]

It works for analogous integral equations, as well. In many cases, functions satisfying such equations approximately generate the exact solutions to them (see, e.g., [2]). That method can be described in the terms of fixed points in suitable function spaces (for related results see, e.g., [1, 5, 6]). Some examples of simple applications of it are provided.

References

Present permanent employment: Department of Mathematics, Pedagogical University, Kraków, Poland; position of professor

1983 – Master of Science in Mathematics, Jagiellonian University, Kraków, Poland
1991 – PhD in Mathematics
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Major research interests: functional equations and inequalities with their applications, Ulam’s type stability (e.g., of difference, differential, functional, integral and operator equations), real and functional analysis, fixed point theory.

Author of over 100 papers that are already printed or accepted for publication.

Chairman of the Scientific Committee of the series of conferences: International Conference on Functional Equations and Inequalities (ICFEI) (http://uatacz.up.krakow.pl/icfei/15ICFEI/)


Chairman of the Scientific and Organizing Committees of the conference: Conference on Ulam’s Type Stability, Ustroń (Poland), June 2-6, 2014 (http://cuts.up.krakow.pl/)

Member of the Program Committee of the conference: The 2014 International Conference on Pure Mathematics - Applied Mathematics, Venice, Italy, March 15-17, 2014 (http://tinyurl.com/pm-am2014)

Member of the Scientific Committee of the 1st WSEAS International Conference on Pure Mathematics (PUMA '14), Tenerife, Spain, January 10-12, 2014 (http://www.wseas.org/cms.action?id=7086)

Member of the Scientific Committee of the International Conference on Functional Equations, Geometric Functions and Applications (ICFGA 12), Payame Noor University, Tabriz, Iran, May 10-12, 2012

Editor (jointly with Th.M. Rassias) of the monograph Functional Equations in Mathematical Analysis (nearly 750 pages; collection of 47 papers of 67 authors), volume 52 (2013) of Springer Optimization and Its Applications series, dedicated to the 100th anniversary of S.M. Ulam

Lead Editor of Banach Center Publications volume 99 (2013) titled: Recent Developments in Functional Equations and Inequalities. Selected Topics

Lead Guest Editor of Abstract and Applied Analysis annual special issues: Ulam’s Type Stability (http://www.hindawi.com/journals/aaa/type.stability/) in the years 2012, 2013, 2014

Lead Guest Editor of Journal of Function Spaces (formerly: Journal of Function Spaces and Applications) special issue: Ulam’s Type Stability and Fixed Points Methods (http://www.hindawi.com/journals/jfs/si/329604/cfp/)

Lead Guest Editor of Discrete Dynamics in Nature and Society special issue: Approximate and Iterative Methods (http://www.hindawi.com/journals/ddns/si/473241/)

Supervisor of four promoted PhD students.

Editor of several international journals.