On some topics in the Ulam stability

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The following question arises naturally while applying mathematical and computational methods in science and engineering: what errors we commit when we replace functions that satisfy some equations only approximately by the exact solutions to those equations. Quite effective tools to evaluate such errors are provided in the theory of Ulam stability (also often called Hyers–Ulam stability). It has been motivated by a question posed by S.M. Ulam in 1940 (see [2, 4, 14, 15]) and concerning approximate homomorphisms of metric groups. At present, it is a large field of research and concerns various types of equations (e.g., difference, differential, functional, integral). Roughly speaking, nowadays, we understand such stability of an equation in the following way:

When a function satisfying an equation approximately (in some sense) must be near an exact solution to the equation?

The talk contains some basic definitions and examples of (early and recent) results, concerning mainly difference, differential, and integral equations (see, e.g., [1]–[17]). For instance, the following definition describes the main ideas of the notions of stability and hyperstability (see [4]).

Definition 1. *Let A be a nonempty set, (X,d) be a metric space, $\mathcal{E}\subset\mathcal{C}\subset{\mathbb{R}_+}^{A^n}$ be nonempty, $\mathcal{T}$ be an operator mapping $\mathcal{C}$ into ${\mathbb{R}_+}^{A}$ and $\mathcal{F}_1,\mathcal{F}_2$ be operators mapping a nonempty set $\mathcal{D}\subset X^A$ into X^{A^n}.*

We say that the equation

$$\mathcal{F}_1\varphi(x_1,\dots,x_n)=\mathcal{F}_2\varphi(x_1,\dots,x_n) \tag{1}$$

is $(\mathcal{E},\mathcal{T})$ –stable provided for any $\varepsilon\in\mathcal{E}$ and $\varphi_0\in\mathcal{D}$ with

$$d\big(\mathcal{F}_1\varphi_0(x_1,\dots,x_n),\mathcal{F}_2\varphi_0(x_1,\dots,x_n)\big)\le\varepsilon(x_1,\dots,x_n),\qquad x_1,\dots,x_n\in A, \tag{2}$$

there exists a solution $\varphi\in\mathcal{D}$ of equation (1) such that

$$d\big(\varphi(x),\varphi_0(x)\big)\le\mathcal{T}\varepsilon(x),\qquad x\in A. \tag{3}$$

Next, given $\varepsilon\in{\mathbb{R}_+}^{A^n}$, we say that equation (1) is ε –hyperstable provided every $\varphi_0\in\mathcal{D}$, satisfying (2), fulfils equation (1).

The notion of non-stability is discussed, as well, on the examples of the difference equation of the form

$$x_{n+1}=F(x_n),$$

and its generalizations (also of higher orders).

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