Bipartite theory of Semigraphs

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Abstract: - Given a semigraph, we can construct graphs S_a, S_{ca}, S_e and S_{1e}. In the same pattern, we construct bipartite graphs CA(S), A(S), VE(S), $CA^+(S)$ and $A^+(S)$. We find the equality of domination parameters in the bipartite graphs constructed with the domination and total domination parameters of the graphs S_a and S_{ca}. We introduce the domination and independence parameters for the bipartite semigraph. We have defined Xa-chromatic number, Xa-hyperindependent number and Xa-irredundant number. Using these parameters, we have defined a Xa-dominating sequence chain.

Key-Words: - Semigraph, Xa-dominating set, Ya-dominating set, Xa-independent set, Xa-hyperindependent set, hyper Xa-independent set.

1 Introduction

Semigraphs introduced by E. Sampathkumar [6] is an interesting type of generalization of the concept of graph. The semigraphs come closer to graphs in some sense than hyper graphs.

Road networks can be easily modeled by using semigraphs. Traffic routing and density of traffic in junctions may be studied through domination in semigraphs. S. S Kamath and Bhat [5] introduced adjacency domination in semi graphs. S. S. Kamath and Saroja R. Hebber [4] introduced strong and weak domination in semigraphs. Various types of domination like ca-domination, edge domination, weak e-domination, e-domination, ev-domination and (m,e)-strong domination was introduced by Gomathi [1].

In graphs, Hedetienemi and Renulaskar [7, 8] introduced bipartite theory of graphs in which given an arbitrary graph, bipartite graphs were constructed from the given graph which faithfully represents the given graph. In a similar way, we give the bipartite theory of semigraphs. Given a semigraph, we construct bipartite graphs which represent the arbitrary semigraph.

2 Preliminaries

We give the definitions as in [6].

Definition 2.1: *A semigraph S is a pair (V,X) where V is a nonempty set whose elements are called vertices of S, and X is a set of ordered n-tuples, called edges of S, of distinct vertices, for various $n\geq 2$, satisfying the following conditions:*

SG1: Any two edges have at most one vertex in common.

SG2: Two edges $E_1 = (u_1, u_2 \dots u_m)$ and $E_2= (v_1, v_2 \dots v_n)$ are considered to be equal if and only if

1. $m = n$ and

2. Either $u_i = v_i$ for $1\leq i \leq n$ or $u_i = v_{n-i+1}$ for $1\leq i \leq n$.

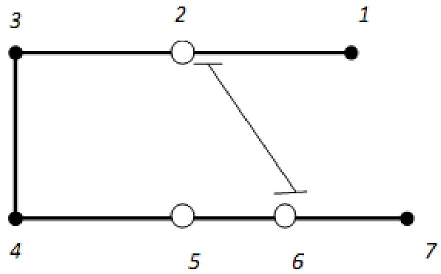
Thus the edge $(u_1, u_2 \dots u_m)$ is the same as $(u_m, u_{m-1} \dots u_1)$. u_1 and u_m are said to be the end vertices of the edge E_1 while $u_2,u_3,\dots,u_{m-1}$ are said to be the middle vertices of E_1.

Note 2.2: *The vertices in a semigraph are divided into four types namely end vertices, middle vertices, middle-end vertices and isolated vertices.*

A semigraph S may be drawn as a set of points representing the vertices. An edge $E = (v_{i1},v_{i2},\dots,v_{ir})$ is represented by a Jordan curve joining the points corresponding to the vertices $(v_{i1},v_{i2},\dots,v_{ir})$ in the same order as they appear in E. The end points of the curve (i.e. the end vertices of E) are denoted by thick dots. The points lying in between the end points (i.e. middle vertices of E) are denoted by small circles. If an end vertex v of an edge E is a middle vertex of some edge E^1, a small tangent is drawn to the circle (representing v on E^1) at the end of E.

Example 2.3:

Let S = (V,X) be a semigraph where V = {1,2,3,4,5,6,7} and X={(1,2,3),(2,6),(3,4),(4,5,6,7)}. In S, 1,3,4,7 are end vertices, 5 is middle vertex and 2, 6 are middle-end vertices .The vertex 2 is middle vertex in (1,2,3) and end vertex in (2,6).The vertex 6 is middle vertex in the edge (4,5,6,7) and end vertex in (2,6). Hence, the edge (2,6) is joined by an edge with a small tangent drawn to the circle at vertex 2 and 6.



2.1 Adjacency of two vertices in a semigraph

There are different types of adjacency of two vertices in a semigraph.

1. Two vertices u and v in a semigraph are said to be adjacent if they belong to the same edge.
2. Two vertices u and v are said to be consecutively adjacent if in addition they are consecutive in order as well.
3. Two vertices u and v are said to be e-adjacent if they are the end vertices of edge in the semigraph.
4. Two vertices u and v are said to be 1e-adjacent if both the vertices u and v belong to the same edge and at least one of them is an end vertex of that edge.

Example 2.4: In the semigraph used in Example 2.3 the vertices 4 and 6 are adjacent. Vertices 5 and 6 are consecutively adjacent. Vertices 4 and 7 are e-adjacent. Vertices 5 and 7 are 1e-adjacent.

2.2 Graphs associated with a given semigraph

Example 2.5: Let S = (V, X) be a given semigraph. Following are the four graphs associated with S, each having the same vertex set as that of S:

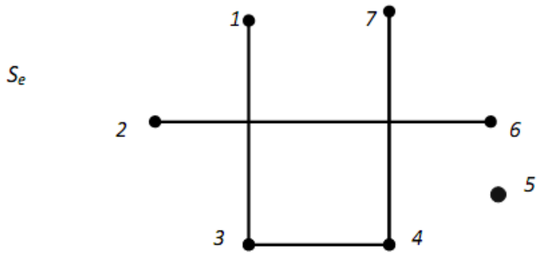
a) **End Vertex graph S_e:** Two vertices in S_e are adjacent if they are the end vertices of an edge in S.

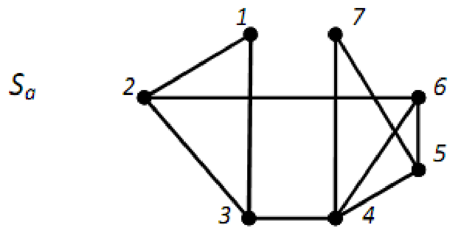
b) **Adjacency graph S_a:** Two vertices in S_a are adjacent if they are adjacent in S.

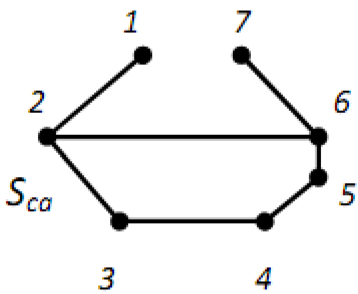
c) **Consecutive adjacency graph S_{ca}:** Two vertices in S_{ca} are adjacent if they are the consecutively adjacent in S.

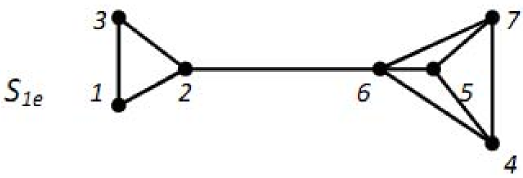
d) **One end Vertex graph S_{1e}:** Two vertices in S_{1e} are adjacent if one of them is an end vertex in S of an edge containing the two vertices.

Example 2.6: The various graph associated with the semigraph given in Example 2.3 is given below









3 Bipartite graphs associated with semigraph

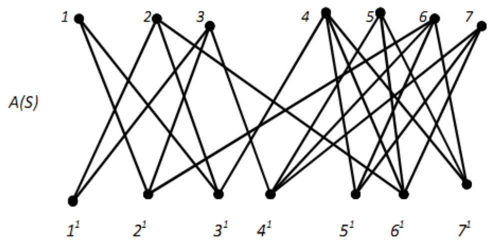
Let V^1 be another copy of the vertex set V of a semigraph. We give below some bipartite graph construction from a given semigraph

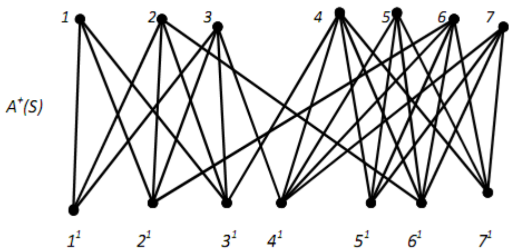
Bipartite graph A(S): The bipartite graph $A(S)=(V,V^1,E)$ where $E=\{(u,v^1)$: u and v belong to the same edge of the semigraph S$\}$.

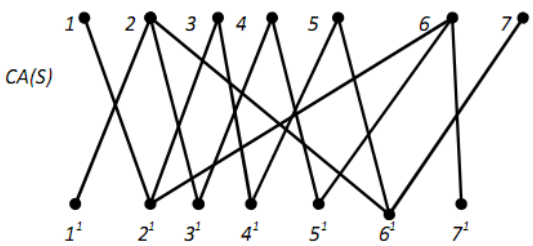
Bipartite graph $A^+(S)$: The graph $A^+(S)=(V,V^1,E^+)$ where E^+ is the set of edges E of the bipartite graph A(S) together with the edges $\{(u,u^1)$: u is in V$\}$.

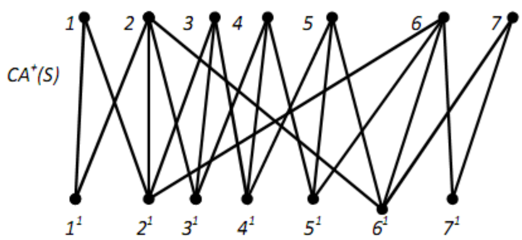
Bipartite graph CA(S): The graph CA(S)=(V,V^1,F) where F={(u,v^1) : u and v are consecutively adjacent in S}.
Bipartite graph CA$^+$(S): The graph CA$^+$(S)=(V,V^1,F^1) where F^1 is the set of edges F of CA(S) together with the edges {(u,u^1) : u is in V}.
Bipartite graph VE(S): The graph VE(S)=(V,E,F) where V is the edge set and E is the edge set of the semigraph S. F={(u,e) : u is a vertex in E}.

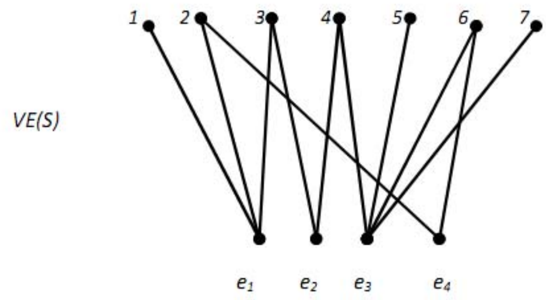
Example 3.1: We give the bipartite constructions of the semi graph given in example 2.3.
We take a copy of vertex set V denoted as V^1={1^1,2^1,3^1,4^1,5^1,6^1,7^1}











4 Dominating sets

We give the definition of various domination parameters for an arbitrary graph and semigraph.

4.1 Dominating sets in graphs

Let G=(V,E) be an arbitrary graph.

Definition 4.1: *[1,2] A subset S of V is a dominating set of a graph G if for every u in V-S, there exists a vertex v in S such that u and v are adjacent.*

The minimum cardinality of a dominating set of a graph G is called the domination number of G and is denoted by γ (G).

Definition 4.2:*[1,2] A subset D of V is a total dominating set of a graph G if for every u in V, there exists v in D such that u and v are adjacent.*

The minimum cardinality of a total dominating set of a graph G is called the total domination number of G and is denoted by γ_t (G).

4.2 Bipartite domination

In the bipartite theory of graphs, various parameters like X-dominating sets, Y-dominating sets introduced. We give the definition of Y-dominating set. Bipartite domination was studied in detail in [7,8].

Definition 4.3: *[7,8]Let G=(X, Y, E) be a bipartite graph. A subset S of X is a Y-dominating set if every element y in Y is adjacent to a vertex of S.*

The minimum cardinality of a Y-dominating set of a graph G is called Y-domination number of G and is denoted by γ_Y (G).

Definition 4.4: *[7,8,9] Let G=(X,Y,E) be a bipartite graph. Two vertices u and v in X are X-adjacent, if they are adjacent to a vertex y in Y.*

A subset D of X is a X-dominating set if for every u in X-D there exists a vertex v in D such that u and v are X-adjacent. The minimum cardinality of a X-

dominating set of a graph G is called the X-domination number of G and is denoted by $\gamma_X(G)$.

4.3 Dominating sets in Semigraphs

We give below definition of some dominating parameters in semigraph.

Let S = (V, X) be a semigraph.

Definition 4.5: *[1] A subset D of V is called a-dominating set of S if for every v in V-D there exists u in D such that u and v are adjacent.*

The minimum cardinality of an a-dominating set of S is called the a-domination number of S and is denoted by $\gamma_a(S)$.

Remark 4.6: $\gamma_a(S) = \gamma(S_a)$.

Definition 4.7: *[1] A subset D of V(S) is called ca-dominating set if for every vertex v in V-D, there exists u in D such that u and v are consecutive vertices of an edge.*

The minimum cardinality of a ca-dominating set is called the ca-domination number of S, denoted by $\gamma_{ca}(S)$.

Remark 4.8: $\gamma_{ca}(S) = \gamma(S_{ca})$.

Theorem 4.9: *For any semigraph S,*

(i) $\gamma_t(S_{ca}) = \gamma_Y(CA(S))$

(ii) $\gamma(S_{ca}) = \gamma_Y(CA^+(S))$.

Proof: **(i)** Let D be a γ_t – set of the graph S_{ca}. For every u in $V(S_{ca})$, there exists v in D such that u and v are adjacent in the graph S_{ca}. In the semigraph S, u and v are consecutively adjacent. In the graph CA(S), u and v^1 are adjacent. Hence, for every v^1 in Y there exists u in D contained in X such that u and v^1 are adjacent. Hence, D is a Y-dominating set of CA(S). Therefore, $\gamma_Y(CA(S)) \leq |D| \leq \gamma_t(S_{ca})$.

Conversely, D^1 be a Y-dominating set in the graph CA(S)= (X,Y,E). For every y in Y there exists x in D^1 such that x and y are adjacent. In the semigraph S, x and y are consecutively adjacent. Therefore, x and y are adjacent in S_{ca}. In the graph S_{ca}, for every y in V, there exists x in D^1, such that x and y are adjacent. Hence, D^1 is a total dominating set in S_{ca}. Hence, $\gamma_t(S_{ca}) \leq |D^1| \leq \gamma_Y(CA(S))$.

(ii) Let D be a γ – set of the graph S_{ca}. For every u in $V(S_{ca})$-D, there exists v in D such that u and v are adjacent in the graph S_{ca}. In the semigraph S, u and v are consecutively adjacent. In the graph $CA^+(S)$, u and v^1 are adjacent. Hence, for every v^1 in Y there exists u in D contained in X such that u and v^1 are adjacent. Hence, D is a Y-dominating set of $CA^+(S)$. Therefore, $\gamma_Y(CA^+(S)) \leq |D| \leq \gamma(S_{ca})$.

Conversely, D^1 be a Y-dominating set in the graph $CA^+(S)$ = (X, Y, E). For every y in Y there exists x in D^1 such that x and y are adjacent. In the semigraph S, x and y are consecutively adjacent. Therefore, x and y are adjacent in S_{ca}. In the graph S_{ca}, for every y in V-D^1, there exists x in D^1, such that x and y are adjacent. Therefore, D^1 is a dominating set in S_{ca}. Hence, $\gamma(S_{ca}) \leq |D^1| \leq \gamma_Y(CA^+(S))$. Therefore, $\gamma(S_{ca})=\gamma_Y(CA^+(S))$.

Theorem 4.10: *For any semigraph S,*

(i) $\gamma_t(S_a) = \gamma_Y(A(S))$

(ii) $\gamma(S_a) = \gamma_Y(A^+(S))$.

Proof: **(i)** Let D be a γ_t – set of the graph S_a. For every u in $V(S_a)$, there exists v in D, such that u and v are adjacent in the graph S_a. In the semigraph S, u and v are adjacent. In the graph A(S), u and v^1 are adjacent. Hence, for every v^1 in Y, there exists u in D contained in X, such that u and v^1 are adjacent. Hence, D is a Y-dominating set of A(S). Therefore, $\gamma_Y(A(S)) \leq |D| \leq \gamma_t(S_a)$.

Conversely, D^1 be a Y-dominating set in the graph A(S) = (X,Y,E). For every y in Y, there exists x in D^1, such that x and y are adjacent. In the semigraph S, x and y are adjacent. Therefore, x and y are adjacent in S_{ca}. In the graph S_{ca}, for every y in V, there exists x in D^1, such that x and y are adjacent. Hence, D^1 is a total dominating set in S_a. Hence, $\gamma_t(S_a) \leq |D^1| \leq \gamma_Y(A(S))$.

(ii) Let D be a γ – set of the graph S_a. For every u in $V(S_a)$-D, there exists v in D, such that u and v are adjacent in the graph S_a. In the semigraph S, u and v are adjacent. In the graph $A^+(S)$, u and v^1 are adjacent. Hence, for every v^1 in Y, there exists u in D contained in X, such that u and v^1 are adjacent. Hence, D is a Y-dominating set of $A^+(S)$. Therefore, $\gamma_Y(A^+(S)) \leq |D| \leq \gamma(S_a)$.

Conversely, D^1 be a Y-dominating set in the graph $A^+(S)$= (X,Y,E). For every y in Y, there exists x in D^1, such that x and y are adjacent. In the semigraph S, x and y are adjacent. Therefore, x and y are adjacent in S_a. In the graph S_a, for every y in V-D^1, there exists x in D^1, such that x and y are adjacent. Therefore, D^1 is a dominating set in S_a. Hence, $\gamma(S_a) \leq |D^1| \leq \gamma_Y(A^+(S))$. Therefore, $\gamma(S_a)=\gamma_Y(A^+(S))$.

Theorem 4.11: *For any semigraph S,*

(i) $\gamma(S_a) = \gamma_X(VE(S))$.

Proof: **(i)** Let D be a γ – set of the graph S_a. For every u in $V(S_a)$, there exists v in D, such that u and v are adjacent in the graph S_a. In the semigraph S, u and v are adjacent. Equivalently, u and v belong to the same edge. In the graph VE(S), u and v are adjacent to an edge e. Hence, for every v in X-D, there exists u in D contained in X, such that u and v are X-adjacent. Hence, D is a X-dominating set of VE(S). Therefore, γ_X (VE(S))≤ |D| ≤γ (S_a).

Conversely, D^1 be a X-dominating set in the graph VE(S) = (X,Y,E). For every y in V-D, there exists x in D^1 , such that x and y are X- adjacent. In the semigraph S, x and y belong to the same edge.Hence, x and y are adjacent. Therefore, x and y are adjacent in S_a. In the graph S_a, for every y in V-D^1, there exists x in D^1 , such that x and y are adjacent. Hence, D^1 is a dominating set in S_a. Hence, γ (S_a) ≤ | D^1| ≤ γ_X (VE(S)).

5 Independent sets

We give the definition of independent sets of arbitrary graph and semigraph.

5.1 Independent sets in graphs

Let G=(V, E) be a graph.

Definition 5.1: *Let S be a subset of V. The set S is called an independent set if any two adjacent vertices of G is not in S.*
The maximum cardinality of an independent set is called the independence number of the Graph G and is denoted by $\beta_0(G)$.

5.2 Independent sets in Bipartite graphs

Let G=(X,Y,E) be a bipartite graph.

Definition 5.2: *[7,8] A subset S of X is called a X-independent set, if no two vertices in S are X-adjacent.*
The maximum cardinality of a X-independent set of a graph G is called the X-independence number of G and is denoted by $\beta_X(G)$.

Definition 5.3: *[7,8]A subset S of X is called a hyper independent set, if for every y in Y, N(y) is not contained in S.*
The maximum cardinality of a hyper independent set of a graph G is called the hyper independence number of G and is denoted by $\beta_h(G)$.

5.2 Independent sets in semigraphs

Let G=(V,X) be a semigraph.

Definition 5.4: *A set S of vertices in a semigraph is independent if no edge is a subset of S.*
The maximum cardinality of an independent set of a semigraph G is called the independence number of G and is denoted by $\beta_i(G)$.

Theorem 5.5: *For any semigraph S,*
(i) $\beta_0(S_a)= \beta_X(VE(S))$.

Proof: (i) Let D be a β_0 – set of the graph S_a. Any two vertices in D are not adjacent in S_a. In the semigraph S, any two vertices in D do not belong to the same edge. In the graph VE(S), any two vertices of D are non X-adjacent. Therefore, D is a X-independent set in VE(S). Hence, β_X (VE(S))≥|D| ≥β_0 (S_a).

Conversely, D^1 be a β_X-set of the graph VE(S). Any two vertices in $D^1$are not X-adjacent. In semigraph S, any two vertices in D^1 do not belong to the same edge. In the graph S_a, any two vertices in D^1 are not adjacent. Hence, D^1 is a independent set in S_a. Hence, β_0 (S_a) ≥ | D^1 | ≥ β_X (VE(S)). Therefore, β_0 (S)=β_X (VE(S)).

Theorem 5.6: *For any semigraph S,*
(i) $\beta_i(S)= \beta_h(VE(S))$.

Proof: (i) Let D be a β_i – set of the graph S. No edge is a subset of D. In the graph VE(S)=(X,Y,F), for every y in Y, N(y) is not contained in D. Therefore, D is a hyper independent set in VE(S). Hence, β_h (VE(S))≥|D| ≥β_i (S).

Conversely, D^1 be a β_h-set of the graph VE(S). For every y in Y, N(y) is not contained in D^1. In semigraph S, no edge is a subset of D^1 . Hence, D^1 is independent. Hence, β_i(S)≥ | D^1| ≥β_h (VE(S)). Therefore, β_i (S)=β_X (VE(S)).

6 Bipartite domination in semigraph

There are four types of bipartite semigraphs, namely bipartite semigraph, e-bipartite semigraph, strongly bipartite semigraph and edge bipartite semigraph. We consider only bipartite semigraphs.

Definition 6.1: *A semigraph S is bipartite if its vertex set V can be partitioned in to sets {X,Y} such that X and Y are independent.*

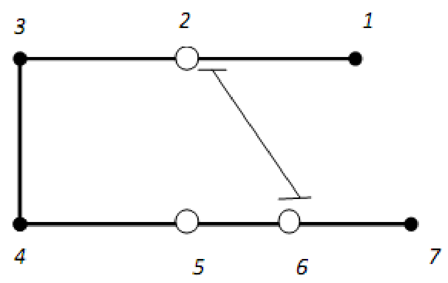
6.1 Dominating sets in semigraphs

Two vertices x in X and y in Y are Ya-adjacent if x and y belongs to the same edge of the semigraph S. Two vertices u and v in X are Xa-adjacent if u and v belongs to the same edge of the semigraph S or an edge E_1 containing u and an edge E_2 containing v are adjacent. Let x belong to X. The set N_{Ya}(x) is the set of vertices Xa-adjacent to x in X.

Definition 6.2: *A subset D of X is called a Ya-dominating set if every vertex y in Y is Ya-adjacent to a vertex of D. The minimum cardinality of a Ya-dominating set is called the Ya-domination number of S and is denoted by $\gamma_Ya(G)$.*

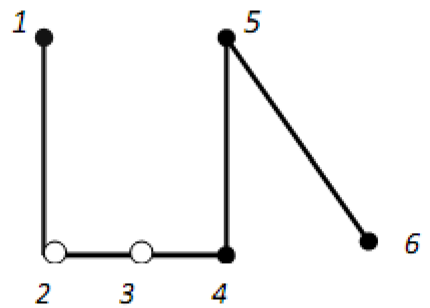
Definition 6.3: *A subset D of X is called a Xa-dominating set if every vertex u in X-D is Xa-adjacent to a vertex of D.*
A subset D of X is called a minimal Xa-dominating set if no proper subset of D is a Xa-dominating set. The minimum cardinality of a minimal Xa-dominating set is called the Xa-domination number of semigraph and is denoted by $\gamma_X a(G)$. The maximum cardinality of a minimal Xa-dominating set is called the upper Xa-domination number of semigraph and is denoted by $\Gamma_X a(G)$.

Example 6.4: Consider the bipartite semigraph



Here X={2,3,5,7} and Y={1,6,4}
D={2,3} is a Ya-dominating set and D_1={3} is a X-dominating set.

Proposition 6.5: *In a semigraph G with no isolates, every Ya-dominating set is a Xa-dominating set.*
Proof: Let D be a Ya-dominating set. Every vertex in Y is Ya-adjacent to a vertex of D. Equivalently, every vertex in Y and a vertex of D belongs to a same edge of the semigraph S. Let x in X-D. Since, G has no isolated vertex, x in X-D is adjacent to a vertex in Y or X. That is x and u in D belong to the same edge of S or an edge E_1 containing x and an edge E_2 containing u are adjacent. Hence, D is a Xa-dominating set.
Remark 6.7: Converse of the above proposition need not be true.
Consider the semi graph given below



{1} is a Xa-dominating set but not Ya-dominating set.
Remark 6. : In a semigraph, $\gamma_X a(G) \leq \gamma_Y a(G)$.

Definition 6.8: *Let S be a subset of X. Let u belong to D. The vertex u is called an Ya-isolate of D if there exists no vertex v in D-{u} such that u and v are Xa-adjacent.*

Theorem 6.9: *A X-dominating set D of a bipartite semigraph is minimal if and only if for every u in D one of the following holds:*
(i) u is a Ya-isolate of D.
(ii) There exists a vertex v in X-D such that $N_{Ya}(v) \cap D = \{u\}$.
Proof: Let D be a minimal Xa-dominating set of G and let u belong to D. Then, D-{u} is not a Xa-dominating set. Hence, some vertex v in X-(D-{u}) is not Xa-adjacent to any vertex in D-{u}. Then either v=u in which case u is Ya-isolate of D which is condition (i) or v belongs to X-D and v is not Xa-adjacent to any vertex of X-(D-{u}). Equivalently, $N_{Ya}(v) \cap D = \{u\}$, which is (ii).
Let us assume that D is not a minimal Xa-dominating set. There exists a vertex v in D such that D-{u} is a Xa-dominating set. Hence, u is Xa-adjacent to at least one vertex in D-{u}, and so condition (i) does not hold for D. Also every vertex in X-D is Xa-adjacent to at least one vertex in D-{u}, and so condition (ii) deos not hold for u. Thus D does not satisfy (i) and (ii).

Theorem 6.10: *Let S be a semigraph with no Xa-isolates. If D is a minimal Xa-dominating set for S, then X-D is an Xa-dominating set for S.*
Proof: Let u belong to D. Since, D is a minimal Xa-dominating set, either u is an Xa-isolate of D or there exists v in X-D such that $N_{Ya}(v) \cap D = \{u\}$. Therefore, u is Xa-adjacent to a vertex v in X-D. Therefore, u is Xa-dominated by X-D. This is true for every u in D. Therefore, X-D is an Xa-dominating set for S.

Corollary 6.11: *If the bipartite semigraph S has no Xa-isolates, then $\gamma_X a(G) \leq (n/2)$, n is the number of vertices in X.*
Proof: Let D be a Xa-dominating set of a bipartite semigraph. Then, by Theorem 6.10, X-D is a Xa-dominating set. Therefore, $\gamma_X a(G) \leq |X\text{-}D|$. Hence, $\gamma_X a(G) \leq (n/2)$.

6.2 Independent sets in semigraphs

Definition 6.12: *Two vertices u and v in X are Xa-independent if u and v are not Xa-adjacent. A* *subset D of X is called a Xa-independent set if any two vertices in D are Xa-independent. A set D is called a maximal Xa-independent set if we cannot find a Xa-independent set D_1 containing D. The maximum cardinality of a maximal Xa-independent set is called the Xa-independence number and is denoted by $\beta_X a(G)$.*

Theorem 6.13: *In a semigraph S, every maximal Xa-independent set is a minimal Xa-dominating set.*
Proof: Let I be a maximal Xa-independent set of X. Let v in V-I. Since, I is a maximal Xa-independent set, there exists an edge containing v and a vertex of I. Therefore, v is Xa-dominated by I. Therefore, I is a Xa-dominating set of the semigraph S.

Definition 6.14: *A subset S of X which is Xa-independent and Xa-dominating is called Xa-independent Xa-dominating set.*

The existence of such a set is guaranteed by the above theorem.

Definition 6.15: *A Xa-independent, Xa-dominating set of minimum cardinality is called Xa-independent, Xa-domination number of a semigraph G and is denoted by $i_X a(G)$.*

Clearly $i_X a(G) \le \beta_X a(G)$. Thus we have

$$\gamma_X a(G) \le i_X a(G) \le \beta_X a(G) \le \Gamma_X a(G).$$

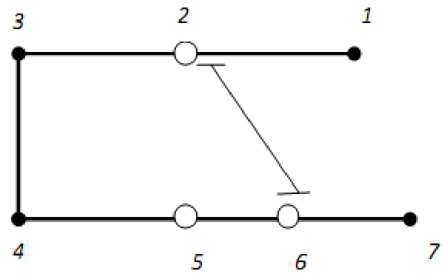
Theorem 6.16: *Let S be a semigraph with $N_{Ya}(x) \neq \varphi$. Then, $\gamma_X a(G) \le |X| - \beta_X a(G)$.*
Proof: Let D be a maximum Xa-independent set of a semigraph S. Then every vertex in D is Xa-adjacent to at least one vertex of X-D. Therefore, X-D is a Xa-dominating set of S. Hence, $\gamma_X a(G) \le |X| - \beta_X a(G)$.

The set $N_{Xa}(y)$ is the set of vertices in X, Ya-adjacent to y in Y.

Definition 6.17: *A subset D of X is called Xa-hyper independent set if $N_{Xa}(y)$ is not contained in D, for every y in Y. The maximum cardinality of a Xa-hyper independent set of a semigraph is called Xa-hyper independence number and is denoted by $\beta_h a(G)$.*

Definition 6.18: *A subset D of X is called hyper Xa-independent set if $N_{Ya}(x)$ is not contained in D, for every x in D. The maximum cardinality of a hyper Xa-independent set of a semigraph is called hyper Xa-independence number and is denoted by $\beta_{hX} a(G)$.*

Example 6.19: Consider the bipartite semigraph



Here X={2,3,5,7} and Y={1,6,4}
D={5,7} is a Xa-hyper independent set and D_1={2,5,7} is a hyper Xa-independent set.

Theorem 6.20: *In a semigraph, every Xa-independent set is a Xa-hyper independent set.*

Theorem 6.21: *In a semigraph, every Xa-hyper independent set is hyper Xa-independent set.*

Theorem 6.22: *In a semigraph, a subset D is a Xa-dominating set if and only if X-D is hyper Xa-independent set.*
Proof: Let D be a Xa-dominating set. For every x in X-D there exists u in D such that u and x are Xa-adjacent. For every x in X-D, $N_{Ya}(x)$ is not contained in X-D. Therefore, X-D is hyper Xa-independent set.

Conversely, let D_1 be a hyper Xa-independent set. That is $N_{Ya}(x)$ is not contained in D_1, for every x in D_1. Equivalently, for every u in D_1, there exists x in X-D_1 such that u and x are Xa-adjacent. Therefore, X-D_1 is a Xa-dominating set.

Corollary 6.23: *In a semigraph, $\gamma_X a(G) + \beta_{hX} a(G) = |X|$.*
Proof: Let S by a Xa-dominating set. Then X-S is hyper Xa-independent set. Hence, $\beta_{hX} a(G) \ge |X| - \gamma_X a(G)$. Therefore, $\beta_{hX} a(G) + \gamma_X a(G) \ge |X|$.
Conversely, let S be a maximum hyper Xa-independent set. Then, X-S is a Xa-dominating set. Therefore, $\gamma_X a(G) \le |X| - \beta_{hX} a(G)$. Hence, $\gamma_X a(G) + \beta_{hX} a(G) \le |X|$. Therefore, $\gamma_X a(G) + \beta_{hX} a(G) = |X|$.

Theorem 6.24: *In a semigraph, a subset D is a Ya-dominating set if and only if X-D is Xa-hyper independent set.*
Proof: Let D be a Ya-dominating set. Every y in Y is Ya-adjacent to a vertex of D. For every y in Y $N_{Xa}(y)$ is not contained in X-D. Therefore, X-D is Xa-hyper independent set.

Conversely, let D_1 be a Xa-hyper independent set. That is $N_{Xa}(y)$ is not contained in D_1, for every y in Y. Equivalently, every y in Y is Ya-adjacent to a vertex of X-D. Therefore, X-D is a Ya-dominating set.

Corollary 6.25: *In a semigraph, $\gamma_Y a(G) + \beta_h a(G) = |X|$.*

Proof: Let S by a Ya-dominating set. Then X-S is Xa-hyper independent set. Hence, $\beta_h a(G) \geq |X| - \gamma_Y a(G)$. Therefore, $\beta_h a(G) + \gamma_Y a(G) \geq |X|$.

Conversely, let S be a maximum Xa- hyper independent set. Then, X-S is a Ya-dominating set. Therefore, $\gamma_Y a(G) \leq |X| - \beta_h a(G)$. Hence, $\gamma_Y a(G) + \beta_h a(G) \leq |X|$. Therefore, $\gamma_Y a(G) + \beta_h a(G) = |X|$.

Theorem 6.26: *In a semigraph, $\gamma_X a(G) = \gamma_Y a(G)$ if and only if there exists a $\gamma_X a(G)$-set S such that X-S is Xa-hyper independent.*

Proof: Let S be a $\gamma_X a(G)$-set S such that X-S is Xa-hyper independent. Then S is a Ya-dominating set. Therefore, $\gamma_Y a(G) \leq |S| = \gamma_X a(G)$. We know that $\gamma_X a(G) \leq \gamma_Y a(G)$. Hence, $\gamma_X a(G) = \gamma_Y a(G)$.

Conversely, assume $\gamma_X a(G) = \gamma_Y a(G)$. Let S be a $\gamma_Y a(G)$ set. Then X-S is Xa-hyper independent set. Since, $\gamma_Y a(G) = \gamma_X a(G)$, there exists a $\gamma_X a(G)$-set S such that X-S is Xa-hyper independent set.

7 Colourings in Bipartite semigraph

We now define Xa-chromatic number and Xa-hyper chromatic number of a bipartite semigraph.

Definition 7.1: *A Xa-colouring of a bipartite semigraph is a partition $\{X_1, X_2, \ldots, X_k\}$ of X into Xa-independent sets. The Xa-chromatic number $\chi_X a(G)$ of a bipartite semigraph G is the smallest order of an X-colouring of G.*

Theorem 7.2: *Let G be a bipartite semigraph on $|X|=p$ vertices. Then, $p/\beta_X a(G) \leq \chi_X a(G) \leq p - \beta_X a(G)+1$.*

Proof: Let $\{X_1, X_2, \ldots, X_k\}$ be a Xa-chromatic partition of X(G). Therefore, $p = \sum |X_i| \leq \chi_X(G) \beta_X a(G)$. Hence, $p/\beta_X a(G) \leq \chi_X a(G)$. Let D be a $\beta_X a(G)$-set of G. Let $D = \{x_1, x_2, \ldots, x_{\beta Xa}\}$. Then $\Pi = \{D, \{x_{\beta Xa+1}\}, \ldots, \{x_p\}\}$ is a Xa-chromatic partition of G. Therefore, $\chi_X a(G) \leq p - \beta_X a(G)+1$. Hence, $p/\beta_X a(G) \leq \chi_X a(G) \leq p - \beta_X a(G)+1$.

Definition 7.3: *A Xa-hyper colouring of a graph is a partition of X into Xa-hyper independent sets. The Xa-hyper independent chromatic number $\chi_h a(G)$, is the smallest order of a Xa-hyper independent colouring of G.*

Theorem 7.4: *Let G be a bipartite semigraph on $|X|=p$ vertices. Then, $p/\beta_h a(G) \leq \chi_X a(G) \leq p - \beta_h a(G)+1$.*

Proof: Let $\{X_1, X_2, \ldots, X_k\}$ be a Xa-hyper independent chromatic partition of X(G). Therefore, $p = \sum |X_i| \leq \chi_h(G) \beta_h a(G)$. Hence, $p/\beta_h a(G) \leq \chi_h a(G)$. Let D be a $\beta_h a(G)$-set of G. Let $D = \{x_1, x_2, \ldots, x_{\beta ha}\}$. Then $\Pi = \{D, \{x_{\beta ha+1}\}, \ldots, \{x_p\}\}$ is a Xa-hyper independent chromatic partition of G. Therefore, $\chi_h a(G) \leq p - \beta_h a(G)+1$. Hence, $p/\beta_h a(G) \leq \chi_h a(G) \leq p - \beta_h a(G)+1$.

8 Xa-Irredundant sets

We now define Xa-irredundant sets and prove the existence of such sets.

Definition 8.1: *Let G be a bipartite semigraph. Let S be a subset of X. Let u belong to S. A vertex v is a private Xa-neighbor of u with respect to S if u is the only vertex of S, Xa-adjacent to v.*

Definition 8.2: *A set S is Xa-irredundant set if every u in S has a private Xa-neighbour. The X-irredundance number of a graph G is the minimum cardinality of a maximal Xa-irredundant set of G and is denoted by $ir_X a(G)$. The upper Xa-irredundance number of a graph G is the maximum cardinality of a maximal Xa-irredundant set of G and is denoted by $IR_X a(G)$.*

Theorem 8.3: *A Xa-dominating set S is a minimal Xa-dominating set if and only if it is Xa-dominating and Xa-irredundant.*

Proof: Let S be a Xa-dominating set. Then S is a minimal Xa-dominating set if and only if for every u in S there exists v in X-(S-{u}) which is not Xa-dominating by S-{u}. Equivalently, S is a minimal Xa-dominating set if and only if for every u in S, u has at least one private Xa-neighbor with respect to S. Thus S is a minimal Xa-dominating set if and only if it is Xa-irredundant.

Conversely, Let S is both Xa-dominating and Xa-irredundant.

Claim: S is a minimal Xa-dominating set.

If S is not a minimal Xa-dominating set, there exists v in S for which S-{v} is Xa-dominating. Since S is Xa-irredundant, v has a private Xa-neighbour with respect to S say u (u may be equal to v). By definition, u is not Xa-adjacent to any vertex in S-{v}. S-{v} is not a xa-dominating set, a contradiction. Hence, S is a minimal Xa-dominating set.

Note: *By the above theorem, any minimal Xa-dominating set is an Xa-irredundant set. Therefore, Xa-irredundant sets exists*.

Theorem 8.4: *Every minimal Xa-dominating set is a maximal Xa-irredundant set.*

Proof: Every minimal Xa-dominating set S is Xa-irredundant set.

Claim: S is a maximal Xa-irredundant set.

Suppose S is not a maximal Xa-irredundant set. Then there exists a vertex u in X-S for which $S\cup\{u\}$ is Xa-irredundant. Therefore, there exists at least one vertex x which is a private Xa-neighbour of u with respect to $S\cup\{u\}$. Hence, no vertex in S is Xa-adjacent to x. Thus S is not Xa-dominating set, a contradiction. Hence, S is maximal Xa-irredundant set.

Remark 8.5: Clearly $ir_X a(G) \leq \gamma_X a(G)$ and $\Gamma_X a(G) \leq IR_X a(G)$.

Thus we have, a Xa-dominating sequence chain,

$$\mathbf{ir_X a(G) \leq \gamma_X a(G) \leq i_X a(G) \leq \beta_X a(G) \leq \Gamma_X a(G) \leq IR_X a(G).}$$

9 Conclusion

We have defined the bipartite graphs, A(S), $A^+(S)$, CA(S), $CA^+(S)$ and VE(S) which faithfully represents the semigraphs. Using the parameters like X-domination number, Y-domination number etc, we have proved the following :

	A(S)	$A^+(S)$	CA(S)	$CA^+(S)$
γ_Y	$\gamma_t(S_a)$	$\gamma(S_a)$	$\gamma_t(S_{ca})$	$\gamma(S_{ca})$

	γ_X	β_h	β_X
VE(S)	$\gamma(S_a)$	$\beta_i(S)$	$\beta_0(S_a)$

We have also introduced bipartite domination for bipartite semigraphs and the parameters like Xa-dominating set, Ya-dominating set, Xa-independent set, Xa-hyper independent set, hyper Xa-independent set, Xa-irredundant sets are introduced. We have characterized semigraphs for which $\gamma_X a(G)= \gamma_Y a(G)$. We have defined Xa-chromatic number and Xa-hyper independent chromatic number and their bounds are also given.

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