New hybrid steepest descent algorithms for variational inequalities over the common fixed points set of infinite nonexpansive mappings

HE SONGNIAN  
Civil Aviation University of China  
College of Science  
Jinbei Road 2898, 300300 Tianjin  
CHINA  
hesongnian2003@yahoo.com.cn

SUN WENWEN  
Civil Aviation University of China  
College of Science  
Jinbei Road 2898, 300300 Tianjin  
CHINA  
sunwenwen2698@126.com

Abstract: New hybrid steepest descent algorithms which are different from Yamada’s hybrid steepest descent algorithms are proposed for solving variational inequalities defined on the common fixed points set of infinite nonexpansive mappings. As the extensions of our main results, algorithms are also given for solving variational inequalities defined on the common fixed points set of infinite $\kappa$-strict pseudo-constractions.

Key–Words: Hilbert space, fixed point, nonexpansive mapping, strict pseudo-contraction, variational inequality, hybrid steepest descent algorithm

1 Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, let $C$ be a nonempty closed convex subset of $H$, and let $F : C \to H$ be a nonlinear operator. We consider the problem of finding a point $x^* \in C$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall \ x \in C. \quad (1)$$

This is known as the variational inequality problem (i.e., $VI(C, F)$), which is introduced initially and studied by Stampacchia [1] in 1964. In recent years, variational inequality problems have been extended to study a large variety of problems arising in structural analysis, economics, optimization, operations research and engineering sciences, see [2-7] and the references therein.

Generally, $F$ is assumed to be strongly monotone and Lipschitzian. Relative definitions are stated as below.

Let $C$ be a nonempty closed and convex subset of a real Hilbert space $H$, $F : C \to H$ and $T : C \to C$, then

(i) $F$ is called Lipschitzian on $C$, if there exists a positive constant $L$ such that

$$\|Fx - Fy\| \leq L\|x - y\|, \quad \forall x, y \in C; \quad (2)$$

(ii) $F$ is called boundedly Lipschitzian on $C$, if for each nonempty bounded subset $B$ of $C$, there exists a positive constant $\kappa_B$ depending only on the set $B$ such that

$$\|Fx - Fy\| \leq \kappa_B\|x - y\|, \quad \forall x, y \in B; \quad (3)$$

(iii) $F$ is said to be $\eta$-strongly monotone on $C$, if there exists a positive constant $\eta$ such that

$$\langle Fx - Fy, x - y \rangle \geq \eta\|x - y\|^2, \quad \forall \ x, y \in C; \quad (4)$$

(iv) $T$ is said to be a $\kappa$-strict pseudo-contraction if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \quad \forall \ x, y \in C.$$

Specially, $T$ is said to be nonexpansive on $C$ if $\kappa = 0$. Obviously, the class of nonexpansive mappings is the proper subclass of strict pseudo-contractions.

The following lemma is known to us.

Lemma 1 Assume that $C$ is a nonempty closed convex subset of a real Hilbert space $H$, $F : C \to H$ is Lipschitzian and strongly monotone, then variational inequality (1) has a unique solution.

Let $T : H \to H$ be a nonexpansive mapping with the set of fixed points $Fix(T) = \{ x \in C : Tx = x \}$ and $F : H \to H$ is $L$-Lipschitzian and $\eta$-strong monotone. Yamada [8] studied the variational inequality problem $VI(Fix(T), F)$ and proposed a hybrid steepest descent algorithm:

$$x_{n+1} = (I - \mu \lambda_n F)Tx_n \quad (5)$$

and proved the strong convergence, where the sequence $\{\lambda_n\} \subset (0, 1)$ and $\mu \in (0, 2\eta/L^2)$.
Theorem 2 (see [8]) Assume that $H$ is a real Hilbert space, $T : H \to H$ is nonexpansive such that $F_{ix}(T) \neq \emptyset$ and $F : H \to H$ is $\eta$-strongly monotone and $L$-Lipschitzian. Fix a constant $\mu \in (0, 2\eta/L^2)$. Assume also that the sequence $\lambda_n \in (0, 1)$ satisfies the conditions:

(i) $\lambda_n \to 0$ ($n \to \infty$);
(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;
(iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, or
\[ \lim_{n \to \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1. \]

Take $x_0 \in H$ arbitrarily and define $\{x_n\}$ by (5), then $\{x_n\}$ converges strongly to the unique solution of $V I(F_{ix}(T), F)$.

Yamada also studied the variational inequality problem defined on the common fixed points set of finite nonexpansive mappings. Let $H$ be a Hilbert space, $T_1 (n = 1, \cdots, N)$ are nonexpansive mappings from $H$ into itself. $F : H \to H$ be $\eta$-strongly monotone and Lipschitzian. Assume $F_{ix}(T_1) = F_{ix}(T_1T_2 \cdots T_N) = F_{ix}(T_2 \cdots T_N T_1) = \cdots = F_{ix}(T_N - 1 \cdots T_1 T_N) = F_{ix}(T_N T_{N-1} \cdots T_2 T_1) \neq \emptyset$. For solving variational inequality problem $V I(\bigcap_{n=1}^{N} F_{ix}(T_n), F)$, Yamada proposed the following cyclic algorithm:

\[
\begin{align*}
&x_1 = T_1x_0 - \lambda_0 \mu F(T_1x_0) \\
&x_2 = T_2x_1 - \lambda_1 \mu F(T_2x_1) \\
&\quad \cdots \\
&x_N = T_N x_{N-1} - \lambda_{N-1} \mu F(T_N x_{N-1}), \\
&x_{N+1} = T_1 x_N - \lambda_N \mu F(T_1 x_N) \\
&\quad \cdots.
\end{align*}
\]

Indeed, the algorithm above can be rewritten as:

\[
x_{n+1} = T_{[n+1]} x_n - \mu \lambda_n F(T_{[n+1]} x_n)
\]  
(6)

where $\mu \in (0, 2\eta/L^2)$ , $\{\lambda_n\} \subset (0, 1)$ and $T_{[n]} = T_{n \bmod N}$, namely, $T_{[n]}$ is one of $T_1, T_2, \ldots, T_N$ circularly. Yamada got the following result:

Theorem 3 (see [8]) If $\{\lambda_n\} \subset (0, 1)$ satisfies the conditions:

(i) $\lambda_n \to 0$ ($n \to \infty$);
(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;
(iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, or
\[ \lim_{n \to \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1. \]

then the sequence $\{x_n\}$ generated by (6) converges strongly to the unique solution $x^* \in V I(\bigcap_{n=1}^{N} F_{ix}(T_n), F)$.

Let $F$ be a boundedly Lipschitzian and strongly monotone operator and $C$ be a closed convex subset of $H$. Songnian He and Hong-Kun Xu [9] obtained the following results:

Theorem 4 (see [9]) Assume that $F : C \to H$ is boundedly Lipschitzian on $C$ (i.e., for each bounded subset $B$ of $C$, $F$ is Lipschitzian on $B$). Assume also that $F$ is $\eta$-strongly monotone on $C$. Then variational inequality (1) has a unique solution $x^* \in C$ such that

\[ \|x^* - u\| \leq \frac{1}{\eta} \|Fu\|, \]
(7)

where $u \in C$ is an arbitrary fixed point.

Songnian He and Hong-Kun Xu [9] also proved that iterative algorithms can be devised to approximate this solution if $F$ is a boundedly Lipschitzian and strongly monotone operator and $C$ is the set of fixed points of a nonexpansive mapping. They invented a hybrid iterative algorithm:

\[
x_{n+1} = T x_n - \lambda_n \mu F(T x_n), \quad n \geq 0.
\]  
(8)

Theorem 5 (see [9]) Assume that $F : H \to H$ is $\eta$-strongly monotone and boundedly Lipschitzian. Fix an $x_0 \in C = F_{ix}(T)$ arbitrarily and let $\bar{C}$ be the closed ball centered at $x_0$ and with radius $2 \|F_{ix}(0)/\eta$ (i.e., $\bar{C} = S(x_0, 2 \left\|F_{ix}(0)/\eta\right\|$). Denote by $\bar{\mu}$ the Lipschitz constant of $F$ on $\bar{C}$, and take a constant $\mu$ satisfying $0 < \mu < \eta/\bar{\mu}^2$. Assume a sequence $\{\lambda_n\}$ in the unit interval $(0, 1)$ satisfies the conditions:

(i) $\lambda_n \to 0$ ($n \to \infty$);
(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;
(iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, or
\[ \lim_{n \to \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1. \]

Suppose that the sequence $\{x_n\}$ is generated by (8), then $\{x_n\}$ converges strongly to the unique solution $x^*$ of $V I(F_{ix}(T), F)$.

Songnian He and Xiao-lan Liang [10] considered $V I(C, F)$ when $F$ is a boundedly Lipschitzian and strongly monotone operator and $C$ is the set of fixed points of a strict pseudo-contraction $T : H \to H$. Fix a point $x_0 \in F_{ix}(T)$ arbitrarily, set $\bar{C} = S(x_0, 2 \left\|F_{ix}(0)/\eta\right\|$. Denote by $\bar{\mu}$ the Lipschitz constant of $F$ on $\bar{C}$. Fix the constant $\mu$ satisfying $0 < \mu < \eta/\bar{\mu}^2$. Assume also that the sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy $\kappa \leq \alpha_n \leq \alpha < 1$ for a constant $\alpha \in (0, 1)$ and $0 < \lambda_n < 1 (n \geq 0)$ respectively. Let $T_{\alpha_n} = \alpha_n I + (1 - \alpha_n) T$ and $T_{\alpha_n}^{\lambda_n} = (I - \mu \lambda_n F)T_{\alpha_n}$, define $\{x_n\}$ by the scheme:

\[
x_{n+1} = T_{\alpha_n}^{\lambda_n} x_n = (I - \mu \lambda_n F)T_{\alpha_n} x_n \quad (n \geq 0).
\]  
(9)
Theorem 6 (see [10]) If the sequences \(\{\lambda_n\}\) and \(\{\alpha_n\}\) satisfy the conditions:

(i) \(\lambda_n \to 0\) \((n \to \infty)\);  
(ii) \(\sum_{n=0}^{\infty} \lambda_n = \infty\);  
(iii) \(\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty\), \(\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty\), or \(\lim_{n \to \infty} \frac{\lambda_{n+1}}{\lambda_n} = 1\), \(\lim_{n \to \infty} \frac{\alpha_{n+1} - \alpha_n}{\alpha_n} = 0\);

then \(\{x_n\}\) generated by (9) converges strongly to the unique solution \(x^*\) of \(VI(\text{Fix}(T), F)\).

Songnian He and Xiao-lan Liang [10] also considered \(VI(C, F)\) when \(F\) is a boundedly Lipschitzian and strongly monotone operator, \(C\) is the set of common fixed points of finite \(\kappa_i\)-strict pseudocontractions \(T_i : H \to H\) \((i = 1, \ldots, N)\). For such a \(C\), they designed the following hybrid iterative algorithm:

For each \(i = 1, \ldots, N\), let

\[
T_{\alpha_i} = \alpha_i I + (1 - \alpha_i)T_i,
\]

where the constant \(\alpha_i\) such that \(\kappa_i < \alpha_i < 1\). They defined the cyclic algorithm as follows:

\[
x_1 = T_{\alpha_1} x_0 - \mu \lambda_0 F(T_{\alpha_1} x_0),
\]

\[
x_2 = T_{\alpha_2} x_1 - \mu \lambda_1 F(T_{\alpha_2} x_1),
\]

\[
\cdots
\]

\[
x_N = T_{\alpha_N} x_{N-1} - \mu \lambda_{N-1} F(T_{\alpha_N} x_{N-1}),
\]

\[
x_{N+1} = T_{\alpha_1} x_N - \mu \lambda_N F(T_{\alpha_1} x_N),
\]

\[
\cdots
\]

Indeed, the algorithm above can be rewritten as:

\[
x_{n+1} = T_{\alpha_n} x_n - \mu \lambda_n F(T_{\alpha_n} x_n),
\]

where \(T_{\alpha_n} = \alpha_n I + (1 - \alpha_n)T_n; T_n = T_{n \mod N}\), namely, \(T_n\) is one of \(T_1, T_2, \ldots, T_N\) circularly.

Theorem 7 (see [10]) If \(\{\lambda_n\} \subset (0, 1)\) satisfies the conditions:

(i) \(\lambda_n \to 0\) \((n \to \infty)\);  
(ii) \(\sum_{n=0}^{\infty} \lambda_n = \infty\);  
(iii) \(\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty\), \(\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty\), or \(\lim_{n \to \infty} \frac{\lambda_{n+1}}{\lambda_n} = 1\), \(\lim_{n \to \infty} \frac{\alpha_{n+1} - \alpha_n}{\alpha_n} = 0\);

then the sequence \(\{x_n\}\) generated by (10) converges strongly to the unique solution \(x^*\) of \(VI(\cap_{n=1}^{\infty} \text{Fix}(T_n), F)\).

In this paper, motivated by the research above, we introduce general iterative algorithms for solving variational inequality problems \(VI(C, F)\) where \(C\) is the set of common fixed points of infinite non-expansive mappings of \(T_n : H \to H\) \((n = 1, 2, \cdots)\) with \(\cap_{n=1}^{\infty} \text{Fix}(T_n) \neq \emptyset\) and \(F : H \to H\) is a \(\eta\)-strongly monotone and Lipschitzian operator or is a \(\eta\)-strongly monotone and boundedly Lipschitzian operator, respectively. For the two cases of \(F\), we will prove their strong convergence respectively. These algorithms are different from Yamada’s hybrid steepest descent algorithms.

In order to deal with some problems involving the common fixed point set of infinite nonexpansive mappings, \(W\)-mapping is often used, see [11-16]. The \(W\)-mapping is defined by

\[
U_{n+1} = I,
\]

\[
U_{n} = \gamma_n T_n U_{n+1} + (1 - \gamma_n) I,
\]

\[
U_{n-1} = \gamma_{n-1} T_{n-1} U_{n} + (1 - \gamma_{n-1}) I,
\]

\[
\cdots
\]

\[
U_{n-k} = \gamma_{n-k} T_{n-k} U_{n-k+1} + (1 - \gamma_{n-k}) I,
\]

\[
\cdots
\]

\[
U_{n-2} = \gamma_2 T_{n-2} U_{n-3} + (1 - \gamma_2) I,
\]

\[
W_n = U_{n-1} = \gamma_1 T_{n-1} U_{n-2} + (1 - \gamma_1) I,
\]

\[
\cdots
\]

Such a mapping \(W_n\) is called a \(W\)-mapping generated by \(T_1, T_2, \ldots, T_n\) and \(\gamma_1, \gamma_2, \ldots, \gamma_n\).

Since \(W\)-mapping contains many composite operations of \(\{T_n\}\), it is complicated and it needs large computational work. In this paper, we will adopt new method for solving fixed point problem defined on the common fixed points set of infinite nonexpansive mappings. If \(\{x_k\}\) \((k = 1, 2, \ldots)\) is a bounded sequence of \(H\) and \(\{\omega_k\} \subset (0, 1)\) such that \(\sum_{k=1}^{\infty} \omega_k = 1\). It is easy to verify that \(\sum_{k=1}^{\infty} \omega_k x_k\) is convergent. Let \(L_n = \sum_{k=1}^{n} \omega_k T_k\) \((n = 1, 2, \ldots)\), where \(S_n = \sum_{k=1}^{n} \omega_k\). We will replace \(W\)-mapping by \(L_n\) to solve fixed point problems defined on the common fixed points set of infinite non-expansive mappings. Because \(L_n\) doesn’t contain many composite operations of \(\{T_n\}\), it needs less computational work and it is simplistic and easy to realize.

In this paper, we define \(\{x_n\}\) by the scheme:

\[
x_{n+1} = \lambda_n (I - \mu F)x_n + (1 - \lambda_n) L_n x_n,
\]

where \(\lambda_n \subset (0, 1)\) and \(\mu\) is a constant. We will prove \(\{x_n\}\) generated by (11) converges strongly to the unique solution \(x^*\) of \(VI(\cap_{n=1}^{\infty} \text{Fix}(T_n), F)\) under some conditions.

We will use the notations:

- \(\rightharpoonup\) for weak convergence and \(\to\) for strong convergence.
- \(\omega\)-limit set of \(\{x_n\}\).

He Songnian, Sun Wenwen
2 Preliminaries

In this section, some lemmas are given which are important to prove our main results. Lemma 8 and Lemma 10 are clearly to us.

**Lemma 8** Let $H$ be a real Hilbert space. The following expressions hold.

(i) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2$, \forall x, y \in H, \forall t \in [0, 1].

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2(y, x + y)$, \forall $x, y \in H$.

**Lemma 9** (see [17, 18]) Assume $\{a_n\}$ is a sequence of non-negative real numbers satisfying

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n + \sigma_n, \ n = 0, 1, 2, \ldots$$

(12)

where $\{\gamma_n\}_{n=0}^\infty \subset (0, 1), \{\delta_n\}_{n=0}^\infty$ and $\{\sigma_n\}_{n=0}^\infty$ satisfy the conditions:

(i) $\sum_{n=1}^\infty \gamma_n = \infty$,

(ii) $\limsup_{n \to \infty} \delta_n \leq 0$,

(iii) $\sum_{n=1}^\infty |\sigma_n| < \infty$,

then $\lim_{n \to \infty} a_n = 0$.

**Lemma 10** Let $H$ be a real Hilbert space, $F : H \to H$ be $\eta$-strongly monotone and L-Lipschitzian. Fix a constant $\mu$ satisfying $0 < \mu < 2\eta/L^2$, then $I - \mu F$ is a contraction with the contraction factor $1 - \tau$, where $\tau = \frac{\eta}{2}\mu(2\eta - \mu L^2)$.

**Lemma 11** (see [19]) Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $T : C \to C$ a nonexpansive mapping. If a sequence $\{x_n\}$ in $C$ such that $x_n \to z$ and $(1 - T)x_n \to y$, then $(1 - T)z = y$.

**Lemma 12** (see [16]) Assume that $T : H \to H$ is a $\kappa$-strict pseudo-contraction, and the constant $\alpha$ satisfies $\kappa \leq \alpha < 1$. Let

$$T_\alpha = \alpha I + (1 - \alpha)T,$$  

(13)

then $T_\alpha$ is nonexpansive and $Fix(T_\alpha) = Fix(T)$.

**Lemma 13** Let $H$ be a real Hilbert space, $\{x_k\} \subset H$ is bounded, then

$$\|\sum_{k=1}^\infty \omega_kx_k\|^2 = \sum_{k=1}^\infty \omega_k\|x_k\|^2 - \sum_{1 \leq k < l < \infty} \omega_k\omega_l\|x_k - x_l\|^2.$$  

(14)

**Proof:** For any positive integer $m$, let $S_m = \sum_{k=1}^m \omega_k$. From Lemma 8 (i) we can get that

$$\|\sum_{k=1}^\infty \omega_kx_k\|^2 = \|\sum_{k=1}^m \omega_kx_k + \sum_{k=m+1}^\infty \omega_kx_k\|^2 - \|\sum_{k=m+1}^\infty \omega_kx_k\|^2 \leq \|\sum_{k=1}^m \omega_kx_k\|^2 + 2\|S_m\|\|\sum_{k=m+1}^\infty \omega_kx_k\|$$

$$+ \|\sum_{k=m+1}^\infty \omega_kx_k\|^2 = \|S_m\|^2 + \|\sum_{k=m+1}^\infty \omega_kx_k\|^2 + R_m$$

$$= S_m^2 + \sum_{k=m+1}^\infty \omega_k\|x_k\|^2/\omega_1\sum_{1 \leq k < l < \infty} \omega_k\omega_l\|x_k - x_l\|^2 + R_m$$

$$= S_m + \sum_{k=m+1}^\infty \omega_k\|x_k\|^2 - \sum_{1 \leq k < l < \infty} \omega_k\omega_l\|x_k - x_l\|^2 + R_m$$

where $R_m = 2\|\sum_{k=1}^m \omega_kx_k, \sum_{k=m+1}^\infty \omega_kx_k\|^2 + \|\sum_{k=m+1}^\infty \omega_kx_k\|^2$. As $\sum_{k=1}^\infty \omega_kx_k$ is convergent, then $\lim_{m \to \infty} R_m = 0$. Thus (14) follows from taking $m \to \infty$.

**Lemma 14** Let $H$ be a real Hilbert space and $T_k : H \to H$ be $k$-strongly monotone mappings with $\cap_{k=1}^\infty Fix(T_k) \neq \emptyset$. Let $T = \sum_{k=1}^\infty \omega_kT_k$ ($k = 1, 2, \ldots$), where $\{\omega_k\} \subset (0, 1)$ such that $\sum_{k=1}^\infty \omega_k = 1$. Then $T$ is a nonexpansive mapping and $Fix(\sum_{k=1}^\infty \omega_kT_k) = \cap_{k=1}^\infty Fix(T_k)$.

**Proof:** Firstly, we have

$$\|Tx - Ty\|^2 = \|\sum_{k=1}^\infty \omega_kT_kx - \sum_{k=1}^\infty \omega_kT_ky\|^2$$

$$= \|\sum_{k=1}^\infty \omega_k(T_kx - T_ky)\|^2 \leq \sum_{k=1}^\infty \omega_k\|T_kx - T_ky\|^2$$

$$= \|x - y\|^2 \ \forall x, y \in H.$$
\[ \|x - u\|^2 = \sum_{k=1}^{\infty} \omega_k T_k x - u \|^2 \]
\[ = \sum_{k=1}^{\infty} \omega_k (T_k x - T_k u) ^2 \]
\[ - \sum_{1 \leq k < l < \infty} \omega_k \omega_l \|T_k x - T_l x\|^2 \]
\[ \leq \sum_{k=1}^{\infty} \omega_k \|x - u\|^2 - \sum_{1 \leq k < l < \infty} \omega_k \omega_l \|T_k x - T_l x\|^2 \]
\[ = \|x - u\|^2 - \sum_{1 \leq k < l < \infty} \omega_k \omega_l \|T_k x - T_l x\|^2. \]

Hence
\[ \sum_{1 \leq k < l < \infty} \omega_k \omega_l \|T_k x - T_l x\|^2 = 0. \]

This implies that \( T_k x = T_l x \) hold for all \( k, l = 1, 2, \cdots \). Therefore, \( \|x - u\| \leq \|x - T_k x\| + \|T_k x - T_l x\| + \|T_l x - x\| \leq 2 \|x - T_k x\| \leq 2 \|x - u\| \).

**Lemma 15** Let \( H \) be a real Hilbert space and \( T_k : H \to H \) be all non-expansive mappings with \( \bigcap_{n=1}^{\infty} \text{Fix}(T_k) \neq \emptyset \). Let \( T = \sum_{k=1}^{\infty} \omega_k T_k \) where \( \{\omega_k\} \subset (0, 1) \) such that \( \sum_{k=1}^{\infty} \omega_k = 1 \). Assume \( L_n = \sum_{k=1}^{n} \omega_k T_k / S_n \), where \( S_n = \sum_{k=1}^{n} \omega_k \). Then \( L_n \) uniformly converges to \( T \) in each bounded subset \( S \) in \( H \).

**Proof:** Notice \( S \) is bounded and \( \{T_k\} \) are all non-expansive mappings, so \( M = \sup_{x \in S, k \geq 1} \|T_k x\| < \infty \). For all \( x \in S \), we have
\[ \|L_n x - T x\| \leq \sum_{k=1}^{n} \omega_k \|T_k x - S_n\| + \sum_{k=n+1}^{\infty} \omega_k \|T_k x\| \]
\[ \leq \sum_{k=1}^{n} \omega_k \|T_k x - S_n\| + \sum_{k=n+1}^{\infty} \omega_k \|T_k x\| \]
\[ \leq M(1 - S_n) / S_n + M \sum_{k=n+1}^{\infty} \omega_k. \]

Observe that \( (1 - S_n) / S_n \to 0 \) and \( \sum_{k=n+1}^{\infty} \omega_k \to 0 \) as \( n \to \infty \), we obtain \( \|L_n x - T x\| \to 0 \) (\( n \to \infty \)).

**Lemma 16** (see[20]) Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Given \( x \in H \) and \( z \in C \), then \( z = P_C x \) (i.e., \( z \) is metric projection of \( x \) on \( C \) and \( z \) satisfies \( \|x - z\| = \inf \{\|x - y\| : \forall y \in C\} \)). If and only if there holds the relation:
\[ \langle x - z, y - z \rangle \leq 0 \quad \forall y \in C. \]

**3 Main Results**

In this section, we always assume that \( \{T_n\} \) \((n = 1, 2, \cdots)\) is a sequence of nonexpansive mappings from \( H \) to itself with \( \bigcap_{n=1}^{\infty} \text{Fix}(T_n) \neq \emptyset \).

Theorem 17 Assume that \( F : H \to H \) is \( \eta \)-strongly monotone and \( \ell \)-Lipschitzian. Fix a constant \( \mu \in (0, 2\eta / L^2) \), \( \{\lambda_n\} \subset (0, 1) \) satisfies the conditions:

(i) \( \lambda_n \to 0 \) (\( n \to \infty \));

(ii) \( \sum_{n=0}^{\infty} \lambda_n = \infty \);

(iii) \( \sum_{n=0}^{\infty} \lambda_{n+1} - \lambda_n < \infty \), or \( \lim_{n \to \infty} \lambda_{n+1} / \lambda_n = 1 \).

Take \( x_0 \in H \) arbitrarily and define \( \{x_n\} \) by (11), then \( \{x_n\} \) converges strongly to the unique solution of \( VI(\bigcap_{n=1}^{\infty} \text{Fix}(T_n), F) \).

**Proof:** We will prove the result by three steps.

**Step 1** Show that \( \{x_n\} \) is bounded. By Lemma 10, \( \mu F \) is a contraction with the coefficient \( 1 - \tau \), where \( \tau = \frac{1}{\mu} (2\eta / \mu L^2) \), Notice \( x^* \in \text{Fix}(T) \) and \( T : H \to H \) is nonexpansive, we have
\[ \|x_{n+1} - x^*\| \leq \|\lambda_n (1 - \mu F)x_n + (1 - \lambda_n) L_n x_n - x^*\| \]
\[ = \|\lambda_n (1 - F x_n - x^*) + (1 - \lambda_n) (L_n x_n - x^*)\| \]
\[ = \|\lambda_n (1 - F x_n - x^*) + (1 - \lambda_n) (L_n x_n - L_n x^*)\|. \]
By the condition (i) and
\[ \max(\lambda - \lambda_n)\|x_n - x^*\| + (1 - \lambda_n)\|L_n x_n - L_n x^*\| \]
\leq \lambda_n\|(I - \mu F)x_n - x^*\| + (1 - \lambda_n)\|x_n - x^*\|
\leq \lambda_n\|(I - \mu F)x_n - x^*\| + \lambda_n\|\mu Fx^*\|
\leq (1 - \lambda_n)\|x_n - x^*\|
\leq \lambda_n(1 - \tau)\|x_n - x^*\| + \lambda_n\|\mu Fx^*\| + (1 - \lambda_n)\|x_n - x^*\|
\leq \max\{|x_n - x^*|, \frac{\mu}{\tau}\|F x^*\|\}.

By induction we get
\[ \|x_{n+1} - x^*\| \leq \max\{|x_0 - x^*|, \frac{\mu}{\tau}\|F x^*\|\} \]
Thus \( \{x_n\} \) is bounded. Since \( L_n (n = 1, 2, \ldots) \) is non-expansive and \( F \) is \( L \)-Lipschitzian, we can get
\[ \|L_n x_n - L_n x^*\| \leq \max\{|x_0 - x^*|, \frac{\mu}{\tau}\|F x^*\|\} \]
and
\[ \|F x_n - F x^*\| \leq L \max\{|x_0 - x^*|, \frac{\mu}{\tau}\|F x^*\|\}, \]
then \( \{L_n x_n\} \) and \( \{F x_n\} \) are also bounded.

**Step 2.** Show that \( \lim_{n \to \infty} \|x_n - T x_n\| = 0 \).

Firstly, we have from scheme (11) that
\[
\|x_{n+1} - L_n x_n\| = \|\lambda_n (I - \mu F)x_n + (1 - \lambda_n) L_n x_n - L_n x_n\|
\leq \|\lambda_n (I - \mu F)x_n - \lambda_n L_n x_n\| + (1 - \lambda_n)\|L_n x_n - L_n x_n\|
\leq \lambda_n\|(I - \mu F)x_n + \lambda_n\|L_n x_n\|.
\]
By the condition (i) and \( \{L_n x_n\} \) and \( \{F x_n\} \) are bounded, we obtain \( \|x_{n+1} - L_n x_n\| \to 0 \) \((n \to \infty)\).

Secondly, we prove \( \lim_{n \to \infty} \sum_{k=1}^{n} |x_n - x_{n-1}| = 0 \).

Let \( M = \sup_{n}\{(I - \mu F)x_n\} + \|L_n x_n - L_n x^*\| < \infty \), it concludes that
\[ \|x_{n+1} - L_n x_n\| = \|\lambda_n (I - \mu F)x_n + (1 - \lambda_n) L_n x_n - L_n x_n - (I - \mu F)(x_n - x_{n-1})\|
\leq \|\lambda_n (I - \mu F)x_n - \lambda_n L_n x_n\| + (1 - \lambda_n)\|L_n x_n - L_n x_n\|
\leq \lambda_n\|(I - \mu F)x_n + \lambda_n\|L_n x_n\|.
\]
Since \( \sum_{n=1}^{\infty} \omega_n / S_n \) is convergent, it is easy to see that \( \sum_{n=1}^{\infty} \|L_n x_n - L_n x_n - x_{n-1}\| \) is also convergent. Using Lemma 9, it follows \( \|x_{n+1} - x_n\| \to 0 \) \((n \to \infty)\).

It is easy to have
\[ \|x_n - T x_n\| = \|x_n - x_{n+1} + x_{n+1} - L_n x_n + L_n x_n - T x_n\|
\leq \|x_{n+1} - x_n\| + \|x_{n+1} - L_n x_n\| + \|L_n x_n - T x_n\|.
\]
By Lemma 15, we obtain \( \|x_n - T x_n\| \to 0 \) \((n \to \infty)\).

It follows from Lemma 11 that \( \omega_n(x_n) \subset \text{Fix}(T) \).

**Step 3.** Show that \( \lim_{n \to \infty} \|x_n - x^*\| = 0 \). By Lemma 8 (ii), we obtain
\[ \|x_{n+1} - x^*\| \]
\[ = \|\lambda_n (I - \mu F)x_n + (1 - \lambda_n) L_n x_n - x^*\| \]
\[ = \|\lambda_n (I - \mu F)x_n - \lambda_n x - x^*\| + (1 - \lambda_n)\|(I - \mu F)x_n - x^*\|
\leq \|\lambda_n (I - \mu F)x_n - \lambda_n x - x^*\| + (1 - \lambda_n)\|\mu F x^*\|
\leq \|\lambda_n (I - \mu F)x_n + (1 - \lambda_n)\|L_n x_n - L_n x^*\|
\leq \lambda_n(1 - \tau)\|x_n - x^*\| + \lambda_n\|\mu F x^*\| + (1 - \lambda_n)\|x_n - x^*\|
\leq \max\{|x_n - x^*|, \frac{\mu}{\tau}\|F x^*\|\}.
\]
\[(L_n x_n - L_n x^\ast)^2 + 2\langle -\lambda_n \mu Fx^\ast, x_{n+1} - x^\ast \rangle \leq \lambda_n (1 - \tau)\|x_n - x^\ast\| + (1 - \lambda_n)\|x_n - x^\ast\| + 2\langle -\lambda_n \mu Fx^\ast, x_{n+1} - x^\ast \rangle \leq (1 - \tau \lambda_n)\|x_n - x^\ast\|^2 + 2\lambda_n \|F x^\ast, x_{n+1} - x^\ast\|.

In fact, there exists a subsequence \(\{x_{n_j}\} \subset \{x_n\}\) such that
\[
\limsup_{n \to \infty} \langle -F x^\ast, x_n - x^\ast \rangle = \lim_{j \to \infty} \langle -F x^\ast, x_{n_j} - x^\ast \rangle.
\]

Without loss of generality, we may further assume that \(x_{n_j} \to \tilde{x} \in Fix(T)\). Since \(x^\ast\) is the unique solution of \(VI(Fix(T), F)\), we obtain
\[
\limsup_{n \to \infty} \langle -F x^\ast, x_n - x^\ast \rangle = \lim_{j \to \infty} \langle -F x^\ast, x_{n_j} - x^\ast \rangle = -\langle F x^\ast, \tilde{x} - x^\ast \rangle \leq 0.
\]

Finally, we conclude that \(\lim_{n \to \infty} \|x_n - x^\ast\| = 0\) from the conditions (i)-(iii) and Lemma 9.

By Theorem 17, we get one algorithm for finding the common fixed point with minimum norm of infinite nonexpansive mappings.

**Corollary 18** Let \(\{x_{n+1}\}\) be determined by the scheme:
\[
x_{n+1} = \lambda_n \gamma x_n + (1 - \lambda_n) L_n x_n \tag{17}
\]
where \(\gamma \in (-1, 1)\) and \(\{\lambda_n\} \subset (0, 1)\) satisfies the conditions:
(i) \(\lambda_n \to 0 (n \to \infty)\);
(ii) \(\sum_{n=0}^{\infty} \lambda_n = \infty\);
(iii) \(\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, or \lim_{n \to \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1\).

Then \(x_n \to P_{\cap_n Fix(T_n)} 0\).

**Proof:** Taking \(F = I\) in (11), we obtain \(L = 1\) and \(\eta = 1\). Fix a constant \(\mu \in (0, 2\eta/L^2) = (0, 2)\), it concludes that \(-1 < 1 - \mu < 1\). Let \(\gamma = 1 - \mu\), then (11) was rewritten into (17). Using Theorem 17, \(\{x_n\}\) converges strongly to the unique solution \(x^\ast\) of \(VI(\cap_n Fix(T_n), F)\), that is
\[
\langle x^\dagger, x - x^\dagger \rangle \geq 0, \quad \forall \; x \in \cap_n Fix(T_n). \tag{18}
\]

Hence
\[
(0 - x^\dagger, x - x^\dagger) \leq 0, \quad \forall \; x \in \cap_n Fix(T_n). \tag{19}
\]

Using Lemma 16 and (19), we have that \(x^\dagger = P_{\cap_n Fix(T_n)} 0\).

Now we turn to discussing \(VI(\cap_{n=1}^{\infty} Fix(T_n), F)\) where \(F\) is a boundedly Lipschitzian and strongly monotone operator. Fix a point \(x_0 \in \cap_{n=1}^{\infty} Fix(T_n)\) arbitrarily, set \(C = S(x_0, 2\|Fx_0\|/\eta)\). Denote by \(\hat{\lambda}\) the Lipschitz constant of \(F\) on \(C\). Fix the constant \(\mu\) satisfying \(0 < \mu < \eta/\hat{\lambda}\). It follows from Theorem 4 that \(VI(\cap_{n=1}^{\infty} Fix(T_n), F)\) has a unique solution \(x^\ast\).

Our second main result is as follow.

**Theorem 19** Assume that \(F: H \to H\) is a boundedly Lipschitzian and strongly monotone operator. Fix a point \(x_0 \in \cap_{n=1}^{\infty} Fix(T_n)\) arbitrarily, set \(\hat{C} = S(x_0, 2\|Fx_0\|/\eta)\). Denote by \(\hat{\lambda}\) the Lipschitz constant of \(F\) on \(C\). Fix a constant \(\mu\) satisfying \(0 < \mu < \eta/\hat{\lambda}\).

Suppose \(\{\lambda_n\} \subset (0, 1)\) satisfies the conditions:
(i) \(\lambda_n \to 0 (n \to \infty)\);
(ii) \(\sum_{n=0}^{\infty} \lambda_n = \infty\);
(iii) \(\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, or \lim_{n \to \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1\).

Take \(x_0 \in \cap_{n=1}^{\infty} Fix(T_n)\) arbitrarily and define \(\{x_n\}\) by (11), then \(\{x_n\}\) converges strongly to the unique solution of \(VI(\cap_{n=1}^{\infty} Fix(T_n), F)\).

**Proof:** We will also divide the proof into three steps.

**Step 1.** We prove that \(x_n \in \hat{C}\) for all \(n \geq 0\) by induction. It is trivial that \(x_0 \in \hat{C}\). Suppose we have proved \(x_n \in \hat{C}\), that is
\[
\|x_n - x_0\| \leq 2\|Fx_0\|/\eta. \tag{20}
\]

By Lemma 10, \(I - \mu F\) is a contraction on \(\hat{C}\) with the contraction coefficient \(1 - \hat{\tau}\), where \(\hat{\tau} = \frac{1}{2}\mu(2\eta - \mu\hat{\lambda}^2)\). Notice \(L_n : H \to H\) is nonexpansive, we have
\[
\|x_{n+1} - x_0\| = \|\lambda_n(I - \mu F)x_n + (1 - \lambda_n)L_n x_n - x_0\|
\]
\[
= \|\lambda_n((I - \mu F)x_n - x_0) + (1 - \lambda_n)(L_n x_n - x_0)\|
\]
\[
= \|\lambda_n((I - \mu F)x_n - x_0) + (1 - \lambda_n)(L_n x_n - L_n x_0)\|
\]
\[
\leq \lambda_n\|((I - \mu F)x_n - x_0)\| + (1 - \lambda_n)\|L_n x_n - L_n x_0\|
\]
\[
\leq \lambda_n\|((I - \mu F)x_n - x_0)\| + (1 - \lambda_n)\|x_n - x_0\|
\]
\[
= \lambda_n\|((I - \mu F)x_n - x_0) + (1 - \mu F)x_0\| + \lambda_n\|F x_0\|
\]
\[
\leq \lambda_n\|((I - \mu F)x_n - x_0)\| + (1 - \lambda_n)\|x_n - x_0\|
\]
\[
\leq \lambda_n\|((I - \mu F)x_n - x_0)\| + (1 - \lambda_n)\|x_n - x_0\|
\]
\[
= (1 - \lambda_n\hat{\tau})\|x_n - x_0\| + \lambda_n\|F x_0\|/\hat{\tau}
\]
\[
\leq \lambda_n\|x_n - x_0\|/\hat{\tau} + \lambda_n\|F x_0\|/\hat{\tau}
\]
\[
\leq \max\{\|x_n - x_0\|/\hat{\tau}, \|F x_0\|/\hat{\tau}\}
\]
\[
\leq \max\{\|x_n - x_0\|, \|F x_0\|\}.
\]
On the other hand, since \( 0 < \mu < \eta / \hat{L}^2 \) and \( \hat{\tau} = \frac{1}{2} \mu (2 \eta - \mu \hat{L}^2) \), we get
\[
\frac{\mu}{\hat{\tau}} = \frac{\mu}{\frac{1}{2} \mu (2 \eta - \mu \hat{L}^2)} = \frac{2}{\eta (\eta - \mu \hat{L}^2)} < \frac{2}{\eta}
\]
this implies
\[
x_{n+1} - x_0 \leq 2 \| Fx_0 \| / \eta.
\]
It implies that \( x_{n+1} \in \hat{C} \). Therefore, \( x_n \in \hat{C} \) for all \( n \geq 0 \) and \( \{x_n\} \) is bounded.

Since \( L_n \) (\( n = 1, 2, \ldots \)) is nonexpansive and \( F \) is \( L \)-Lipschitzian on \( \hat{C} \), we get
\[
\| L_n x_n - L_n x_0 \| \leq \| x_n - x_0 \| \leq 2 \| Fx_0 \| / \eta
\]
and
\[
\| Fx_n - Fx_0 \| \leq \hat{L} \| x_n - x_0 \| \leq 2 \hat{L} \| Fx_0 \| / \eta,
\]
then \( \{L_n x_n\} \) and \( \{Fx_n\} \) are also bounded.

**Step 2.** Show that \( \lim_{n \to \infty} \| x_n - T x_n \| = 0 \). Firstly, we have from scheme (11) that
\[
\| x_{n+1} - L_n x_n \|
= \| \lambda_n (I - \mu F) x_n + (1 - \lambda_n) L_n x_n - L_n x_n \|
\leq \lambda_n \| (I - \mu F) x_n \| + \lambda_n \| L_n x_n \|.
\]
By the condition (i) and \( \{L_n x_n\} \) and \( \{Fx_n\} \) are bounded, we obtain \( \| x_{n+1} - L_n x_n \| \to 0 \) (\( n \to \infty \)). Secondly, we prove \( \lim_{n \to \infty} \| x_{n+1} - x_n \| = 0 \).

Let \( M = \sup_{n \geq 1} \| (I - \mu F) x_n \| + \| L_n x_n \| < \infty \), we can get
\[
\| x_{n+1} - x_n \|
\leq \| \lambda_n (I - \mu F) x_n + (1 - \lambda_n) L_n x_n - x_n \| + \| L_n x_n - x_n \|
\leq \| \lambda_n (I - \mu F) x_n \| + \| (1 - \lambda_n) L_n x_n \| + \| L_n x_n - x_n \|
\leq \| \lambda_n (I - \tau) \| x_n - x_n \| + \| (1 - \lambda_n) \| L_n x_n - x_n \| + \| L_n x_n - x_n \|
\leq (1 - \lambda_n) \| L_n x_n - x_n \| + \| L_n x_n - x_n \|
\leq \| x_{n+1} - x_n \| + \| x_{n+1} - x_n \|.
\]
By the same proof in Theorem 17 , we conclude that \( \| x_{n+1} - x_n \| \to 0 \) (\( n \to \infty \)). By triangle inequality , we obtain \( \| x_n - T x_n \| \to 0 \) (\( n \to \infty \)). It follows from Lemma 11 that \( \omega_n (x_n) \subset Fix(T) \).

**Step 3.** Show that \( \lim_{n \to \infty} \| x_n - x^* \| = 0 \). By Lemma 8(ii), we obtain
\[
\| x_{n+1} - x^* \|^2
= \| \lambda_n (I - \mu F) x_n + (1 - \lambda_n) L_n x_n - x^* \|^2
\leq \| \lambda_n (I - \mu F) x_n - \lambda_n (I - \mu F x^*) - \lambda_n \| F x^* \|^2
\end{equation}
\[
\leq \| \lambda_n (I - \mu F) x_n - \lambda_n (I - \mu F x^*) - (1 - \lambda_n) \| L_n x_n - L_n x^* \|^2
\leq \| \lambda_n (1 - \tau) \| x_n - x^* \| + \| (1 - \lambda_n) \| x_n - x^* \|
\leq (1 - \tau) \| x_n - x^* \|^2 + 2 \mu \lambda_n \| F x^*, x_{n+1} - x^* \|
\leq (1 - \frac{1}{\tau}) \| x_n - x^* \|^2 + 2 \mu \lambda_n \| - F x^*, x_{n+1} - x^* \|
\]
By the same proof in Theorem 17 , we conclude that \( \| x_{n+1} - x^* \| = 0 \).

**Remark 20** In the practical computation, we can get a common fixed point in \( \cap_{i=1}^{\infty} Fix(T_i) \) as the initial iterative point by using any kind of algorithms, for example, the algorithm 17 in Corollary 18 .

Now we apply the two results above to solve variational inequalities defined on the common fixed points set of infinite strict pseudo-contractions.

Let \( \{T_i\} (i = 1, 2, \ldots) \) is a sequence of \( \kappa_i \)-strict pseudo-contractions from \( H \) to itself with \( \cap_{i=1}^{\infty} Fix(T_i) \neq \emptyset \). For any \( T_i \), fix a constant \( \alpha_i \) such that \( \kappa_i < \alpha_i < 1 \). Let \( T_{\alpha_i} = \alpha_i I + (1 - \alpha_i) T_i \) from Lemma 12, \( T_{\alpha_i} \) is a nonexpansive mapping and \( Fix(T_{\alpha_i}) = Fix(T_i) \).

Let \( T \chi = \sum_{i=1}^{\infty} \omega_i T_i \), where \( \omega_i \) satisfy \( \omega_i > 0 \) and \( \sum_{i=1}^{\infty} \omega_i = 1 \). By Lemma 14, we get \( Fix(\sum_{i=1}^{\infty} \omega_i T_i) = \cap_{i=1}^{\infty} Fix(T_i) \) and \( T \) is a nonexpansive mapping. It is easy to get \( \cap_{i=1}^{\infty} Fix(T_i) = \cap_{i=1}^{\infty} Fix(T_{\alpha_i}) \) and \( \cap_{i=1}^{\infty} Fix(T_i) = Fix(T) \). Let \( \hat{L}_n = \sum_{i=1}^{n} \frac{1}{\omega_i} T_i \), where \( S_n = \sum_{i=1}^{n} \omega_i \) (\( n = 1, 2, \ldots \)). By Lemma 15, \( \hat{L}_n \) uniformly converges to \( T \) on every bounded subset \( S \) in \( H \). Thus \( VI(\cap_{i=1}^{\infty} Fix(T_i), F) \) is equivalent to \( VI(Fix(T), F) \).

We define \( \{x_n\} \) by the scheme:
\[
x_{n+1} = \lambda_n (I - \mu F) x_n + (1 - \lambda_n) \hat{L}_n x_n.
\]
By Theorem 17 and Theorem 19, we obtain the following two results respectively.
Theorem 21 Assume that $F : H \to H$ is $\eta$-strongly monotone and $L$-Lipschitzian. Fix a constant $\mu \in (0, 2\eta/L^2)$. \{\lambda_n\} $\subset (0, 1)$ satisfies the conditions:

(i) $\lambda_n \to 0$ \textit{(}n $\to \infty$\textit{)};
(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;
(iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, or
\[ \lim_{n \to \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1. \]

Take $x_0 \in H$ arbitrarily and define \{\{x_n\}\} by (21), then \{\{x_n\}\} converges strongly to the unique solution of $VI(\cap_{n=1}^{\infty} \Fix(T_n), F)$.

Theorem 22 Assume that $F : H \to H$ is a boundedly Lipschitzian and strongly monotone operator. Fix a point $x_0 \in \cap_{n=1}^{\infty} \Fix(T_n)$ arbitrarily, set $\hat{C} = S(x_0, 2||Fx_0||/\eta)$. Denote by $\mathcal{C}$ the Lipschitz constant of $F$ on $\hat{C}$. Fix the constant $\mu$ satisfying $0 < \mu < \eta/L^2$. Suppose \{\lambda_n\} $\subset (0, 1)$ satisfies the conditions:

(i) $\lambda_n \to 0$ \textit{(}n $\to \infty$\textit{)};
(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;
(iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, or
\[ \lim_{n \to \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1. \]

Take $x_0 \in \cap_{n=1}^{\infty} \Fix(T_n)$ arbitrarily and define \{\{x_n\}\} by (21), then \{\{x_n\}\} converges strongly to the unique solution of $VI(\cap_{n=1}^{\infty} \Fix(T_n), F)$.

Acknowledgements: This paper is supported by Fundamental Research Funds for the Central Universities Grant no. ZZH2011C002.

References:


[16] Lu-Chuan Ceng, Jen Chih Yao, Hybrid viscosity approximation schemes for equilibrium.

