Abstract: In this paper, we investigate the maximal and minimal solutions for initial value problem of fourth order impulsive differential equations by using cone theory and the monotone iterative method to some existence results of solution are obtained. As an application, we give an example to illustrate our results.

Key-words: Banach space, Cone, Initial value problem, Impulsive integro-differential equations

1. Introduction

Impulsive integro-differential equations have become more important in recent years in some mathematical models in physics, chemical technology, population dynamics, biotechnology and economics. For an introduction of the basic theory of impulsive differential equations in $R_n$, see [1]. Impulsive integro-differential equations both for first and second order have been studied by many authors, see [4-13]. Only a few papers have implemented the fourth order impulsive equations, see [14-15]. In [14], the author uses variation methods and a three critical points theorem to investigate impulsive differential equations without impulsive differential inequalities.

In this paper, by applying a new corresponding result connected with fourth-order impulsive differential inequalities, we apply cone theory and the monotone iterative method to investigate the maximal and minimal solutions.

Consider the following initial value problem of fourth order impulsive differential equations:

$$
\begin{align*}
I_{1k} & \in C[E \times E, E], \quad I_{3k} \in C[E, E], \\
I_{3k} & \in C[E \times E \times E, E], \quad (k = 1, 2, \ldots, m), \\
(Tx)(t) & = \int_0^1 k(t, s)x(s)ds, \\
(Sx)(t) & = \int_0^a h(t, s)x(s)ds, \\
\forall t \in J, \quad k \in C[D, R_n], \quad D = \{(t, s) \in J \times J \mid t \geq s\}, \\
h & \in C[J \times J, R_n], \quad R_n = [0, +\infty). \\
\Delta x |_{-t_k} & = x(t_k^+) - x(t_k^-), \\
\Delta x' |_{-t_k} & = x'(t_k^+) - x'(t_k^-), \\
x(t_k^+) & \text{ and } x(t_k^-) \text{ denote the right and left limits of } x \text{ at } t_k, \text{ respectively. Similarly, } x'(t_k^+) \text{ and } x'(t_k^-) \text{ denote the right and left limits of } x' \text{ at } t_k, \text{ respectively.}
\end{align*}
$$

In this paper, we investigate the maximal and minimal solutions of initial value problem of fourth order impulsive differential equations without impulsive differential inequalities.
Because \( x''(t_k^-) \) exists, there exists the limit \( x''(t_k^-) \) of (2) as \( \varepsilon \to 0^+ \), and
\[
x''(t_k^-) = x''(t_k^+) + \int_{t^-}^{t^+} x''(s)ds,
\]
\[\forall t_{k-1} < t < t_k - \varepsilon < t_k, \varepsilon > 0 \tag{2}\]

In the same way, we obtain
\[
x''(t_k^+), x'(t_k), x'(t_k^-).
\]

Let \( x'(t_k) = x'(t_k^-) \), \( x''(t_k) = x''(t_k^-) \), \( x''(t_k^-) = x''(t_k^-) \). Obviously, \( x', x'', x''' \in PC[J, E] \). Indeed, \( PC^3[J, E] \) is a Banach space with the respective norm:
\[
\| x \|_{PC^3} = \max \left\{ \| x' \|_{PC}, \| x'' \|_{PC}, \| x''' \|_{PC} \right\}.
\]

Let \( PC^3[J, E] = \{ x \in PC[J, E] \mid x'(t) \) is continuous at \( t \neq t_k \), \( x'(t_k^-) \) and \( x'(t_k^+) \) exist \}. For \( x \in PC^3[J, E] \), similarly, \( x'(t_k^-) \), \( x'(t_k^+) \) exist.

Let \( x'(t_k^-) = x'(t_k^-) \), \( x''(t_k^-) = x''(t_k^-) \). Obviously, \( x', x'' \in PC[J, E] \). Indeed, \( PC^2[J, E] \) is a Banach space with the respective norm:
\[
\| x \|_{PC^2} = \max \left\{ \| x' \|_{PC}, \| x'' \|_{PC} \right\}.
\]

Let \( PC^2[J, E] = \{ x \in PC[J, E] \mid x'(t) \) is continuous at \( t \neq t_k \), \( x'(t_k^-) \) and \( x'(t_k^-) \) exist \}. For \( x \in PC^2[J, E] \), let \( x'(t_k^-) = x'(t_k^-) \). Obviously \( x' \in PC[J, E] \). Indeed, \( PC^1[J, E] \) is a Banach space with respect to the norm:
\[
\| x \|_{PC^1} = \max \left\{ \| x' \|_{PC} \right\}.
\]

Let \( J' = J \setminus \{ t_1, t_2, \ldots, t_m \}, t_0 = 0, t_{m+1} = a, J_0 = [0, t_1], J_1 = (t_1, t_2), \ldots, J_{m-1} = (t_{m-1}, t_m), J_m = (t_m, a] \), \( \tau = \max \{ t_l - t_{l-1} \mid i = 1, 2, \ldots, m+1 \} \). Denote the norm \( \| \cdot \|_{C_1[\tau, \varepsilon]} \) in the space \( C_1[\tau, \varepsilon] \) and denote the norm \( \| \cdot \|_{PC^1[\tau, \varepsilon]} \) in the space \( PC^1[\tau, \varepsilon] \) and \( \| \cdot \|_{PC^2[\tau, \varepsilon]} \) in the space \( PC^2[\tau, \varepsilon] \).

2. Preliminaries
Suppose that \( E \) is a real Banach space which is partially ordered by a cone \( P \subset E \). We say "\( x \leq y \)" if and only if \( y - x \in P \). Moreover \( P \) is called normal if there exists a constant \( N > 0 \) such that for all \( x, y \in E \), \( \theta \leq x \leq y \) implies \( \| x \| \leq N \| y \| \). In the case \( N \) is called the normality constant of \( P \). \( P \) is called regular if there exists \( y \in E \) such that \( x_1 \leq x_2 \leq \cdots \leq x_\ell \leq \cdots \leq y \) implies \( x \in E \) such that \( \| x_n - x \| \to 0 \) as \( n \to \infty \). Further information can be found in [2].

Lemma 2.1. Assume that \( p \in PC^i[\tilde{J}, E] \cap C^2[\tilde{J}', E] \) satisfies
\[
\begin{align*}
\varepsilon' \leq M_1(t), \quad &\forall t \in J, t \neq t_k, \\
\varepsilon' \leq M_2(t), \quad &\forall t \in J, t \neq t_k, \\
\varepsilon' \leq M_3(t), \quad &\forall t \in J, t \neq t_k,
\end{align*}
\]
where \( M_1(t), M_2(t) \) are bounded with \( M_1 \geq 0, M_2 \geq 0 \) on \( J \) and \( M_1, M_2 \in L^1[0, a] \), \( C_i, L_k \) are all nonnegative constants, and we have
(i) \( a(M_1 + a \sum_{k=1}^{m} C_k + M_2^*) + \sum_{k=1}^{m} (L_k (1 + \varepsilon')) \leq 1 \)
(ii) \( M_1 > 0, aM_1 ((e^{aM_1}) - 1) \)
\[
+ e^{aM_1} \sum_{k=1}^{m} (C_k (e^{-L_k}))
\]
\[
+ \sum_{k=1}^{m} (L_k (e^{M_1})) \leq 1 \]
where \( M_1 = \sup \{ M_1(t) \mid t \in J \}, M_2^* = \sup \{ M_2(t) \mid t \in J \}, C_0 = 0, \) then \( \varepsilon' \leq \theta, \varepsilon' \leq \theta, \forall t \in J \).

Proof. Let \( P^* = \{ g \in E^\ast \mid g(x) \geq 0, \forall x \in P \} \). For any \( g \in P^* \) such that \( \varepsilon' = g(p(t)) \), then \( v \in PC^i[\tilde{J}, R] \cap C^2[\tilde{J}', R] \) and \( \varepsilon' = g(p'(t)) \), where \( \varepsilon' = g(p'(t)) \), \( \forall t \in J \). By (3) we have
\[
\begin{align*}
\varepsilon'(t) &\leq -M_1(t)v(t) - M_2(t)v'(t), \forall t \in J, t \neq t_k, \\
\Delta v |_{t_k} = C_k v'(t_k), \\
\Delta v' |_{t_k} &\leq -L_k v(t_k) - L_k v'(t_k), (k = 1, 2, \ldots, m),
\end{align*}
\]
Put
\[
\varepsilon'(0) \leq 0.
\]
\( v_t(t) = v'(t) \) \((t \in J)\), then \( v_t \in PC[J, R] \cap C^1[J', R] \) and 
\[
\begin{align*}
v(t) &= v(0) + \int_0^t v_s(s) ds + \sum_{0 < t_k < t} \Delta v_{|_{t_k}} \\
\ &= v(0) + \int_0^t v_s(s) ds + \sum_{0 < t_k < t} C_k v_t(t_k), \quad \forall t \in J \quad (7)
\end{align*}
\]
so we have by (6)
\[
\begin{align*}
v_t'(t) &\leq M_1(t)(v(0) + \int_0^t v_s(s) ds + \sum_{0 < t_k < t} C_k v_t(t_k)) \\
&\quad - M_2(t)v_t(t), \quad \forall t \in J, t \neq t_k \\
\Delta v_{|_{t_k}} &\leq -L_k(v(0) + \int_0^{t_k} v_s(s) ds) \\
&\quad + \sum_{i=0}^{k-1} C_i v_t(t_k) - L_k v_t(t_k) \\
v_t(0) &\leq v(0) \leq 0, \quad (k = 1, 2, \ldots, m).
\end{align*}
\]
Next, we show
\[
v_t(t) \leq 0, \quad \forall t \in J \quad (9)
\]
We suppose the inequality \( v_t(t) \leq 0, t \in J \) is not true. This means that we can find \( t^* \in J \) such that \( v_t(t^*) > 0 \). We have the next two cases:

Case (a): Assume that \( t^* \in J_j = (t_j, t_{j+1}] \). Let
\[
\inf_{0 \leq t \leq t^*} v_t(t) = -\lambda.
\]
Then \( \lambda \geq 0 \).

(i) \( \lambda = 0 \). By (8), we have
\[
v_t(t) \leq 0, \quad \Delta v_{|_{t_k}} \leq 0.
\]
Then \( v_t(t) \) is decreasing on \([0, t^*]\), so
\[
v_t(t^*) \leq v_t(0) \leq 0.
\]
This is a contradiction with \( v_t(t^*) > 0 \).

(ii) \( \lambda > 0 \). There exists \( t_* \in J_n, n \in \{1, 2, \ldots, m\} \) such that \( v_t(t_*) = -\lambda \) or \( v_t(t_*) = -\lambda \). Below we discuss only the situation when \( v_t(t_*) = -\lambda \). (The proof is similar, when \( v_t(t_*) = -\lambda \). We obtain by (8)
\[
v_t'(t) \leq M_1'(1 + a + \sum_{k=1}^m C_k) \lambda + M_2' \lambda = M_0 \lambda, \quad \forall t \in [0, t^*], t \neq t_k, \quad (10)
\]
\[
\Delta v_{|_{t_k}} \leq L_k(1 + t_k + \sum_{i=0}^{k-1} C_i) \lambda + L_k' \lambda, \quad \forall t_k \leq t^* \quad (11)
\]
where
\[
M_0 = M_1'(1 + a + \sum_{k=1}^m C_k) + M_2', \quad (12)
\]
Then we have
\[
v_t(t^*) - v_t(t_j) = v_t(\xi_j)(t^* - t_j), \quad t_j < \xi_j < t^*,
\]
\[
v_t(t_j) - v_t(t_{j-1}) = v_t(\xi_{j-1})(t_j - t_{j-1}), \quad t_{j-1} < \xi_{j-1} < t_j, \quad \text{for } j = 1, 2, \ldots, m.
\]
\[
v_t(t_{n+1}) - v_t(t_{n+1}) = v_t(\xi_{n+1})(t_{n+1} - t_n), \quad t_n < \xi_{n+1} < t_{n+1}.
\]
By (11) we know
\[
v_t(t_k) = v_t(t_k) + \Delta v_{|_{t_k}} \leq v_t(t_k) + L_k(1 + t_k + \sum_{i=0}^k C_i) \lambda + L_k' \lambda \quad (14)
\]
Combing (10), (11) and (13), (14), this yields
\[
v_t(t^*) - v_t(t_j) \leq L_j(1 + t_j + \sum_{i=0}^{j-1} C_i) \lambda + L_j' \lambda
\]
\[
+ \lambda M_0 (t^* - t_j) \quad (15)
\]
\[
v_t(t_{n+1}) - v_t(t_{n+1}) \leq L_{n+1}(1 + t_{n+1} + \sum_{i=0}^n C_i) \lambda
\]
\[
+ L_{n+1}' \lambda + \lambda M_0 (t_{n+2} - t_{n+1}) \quad (16)
\]
Adding those inequalities, we have
\[
\lambda \leq \lambda M_0 (t_{n+1} - t_n).
\]
This means that
\[
1 < \sum_{k=1}^m L_k(1 + t_k + \sum_{i=0}^{k-1} C_i) + \sum_{k=1}^m L_k + M_0 a. \quad (17)
\]
This is a contradiction with (4).

Case (b): when (ii) satisfies, putting
\[
w(t) = v_t(t) e^{\int_0^t \mu(\xi) d\xi},
\]
by (8) we have
impulsive differential equations:

$$u^*(t) = f(t, (Bu)(t), (Fu)(t), u(t), u'(t), \quad (TBu)(t), (SBu)(t)), \forall t \in J, t \neq t_k,$$

$$\Delta u_{|_{t=t_k}} = I_{2k}(u(t_k)), \quad (k = 1, 2, \ldots, m),$$

$$u(0) = x^*_0, u'(0) = x^*_1$$

where $J = [0, a](a > 0)$, $f \in C[J \times E \times E \times E \times E \times E \times E]$, $0 < t_1 < \cdots < t_k < \cdots < t_m < a$, $I_{2k} \in C[E, E]$, $I_{2k} \in C[E \times E \times E \times E] (k = 1, 2, \ldots, m)$.

$$x^*_1, x^*_2 \in E, (Tu)(t) = \int_{t}^{a} h(t, s)v(s)ds, (Su)(t) = \int_{t}^{a} h(t, s)v(s)ds, \forall t \in J, k \in C[D, R_+], D = \{t, s\}$

Moreover, for any $g \in P^1$, we have $p(t) \leq \theta, \forall t \in J$. This ends the proof.

Lemma 2.2.\textsuperscript{[1]} Let $m \in C[J, R_+]$, $k \in C[D, R_+]$, $\beta_i \geq 0(i = 1, 2, \ldots, m)$ is constant and

$$m(t) \leq \int_{0}^{t} g(t, s)m(s)ds + \sum_{0 \leq i < m} \beta_i m(t_i), \forall t \in J.$$

Then $m(t) \leq 0$.

Lemma 2.3.\textsuperscript{[1]} If $H \subset PC[J, E]$ is a bounded and countable set, then we have $\alpha(H(t)) \in L[J, R_+]$ and

$$\alpha(\int_{0}^{a} x(t) \ dt | x \in H) \leq 2 \int_{0}^{a} \alpha(H(t))dt.$$

Lemma 2.4.\textsuperscript{[1]} Assume that $H \subset PC^1[J, E]$ is bounded and the functions belonging to $H'$ are equicontinuity on $J_k(k = 1, 2, \ldots, m)$.

$$\alpha_{PC^1}(H) = \max \{\sup_{t \in J} \alpha(H(t)), \sup_{t \in J} \alpha(H'(t))\},$$

where $\alpha_{PC^1}$ is a measure of noncompactness in $PC^1[J, E]$.

In order to study the fourth-order impulsive integro-differential equations, we study the second-order impulsive differential equations firstly by method of the reduction of order.

3. Some results of the second order impulsive differential equations

We investigate the following second order
There exist \( M_1(t), M_2(t) \) are bounded with \( M_1 \geq 0, M_2 \geq 0 \) on \( J \) and \( M_1, M_2 \in L^1[0, a], C_k \geq 0, L_0 \geq 0, \) \( (k = 1, 2, \cdots, m) \) such that
\[
f(t, x, y, z, u, v, w) - f(t, x, \tilde{y}, \tilde{z}, \tilde{u}, \tilde{w}) \geq -M_1(t)(z - \bar{z}) - M_2(t)(u - \bar{u}), \quad \forall t \in J,
\]
\[
I_{2k}(u(t)) - I_{2k}(\bar{u}(t)) = C_k(u(t) - \bar{u}(t)),
\]
\[
I_{3k}(x, y, z, u) - I_{3k}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{u}) \geq -L_k(z - \bar{z}) - L_0^k(u - \bar{u}),
\]
\[
(Bu_0)(t) \leq \bar{x} \leq u(t) \leq (Bu_0)(t),
\]
\[
(Fu_0)(t) \leq \bar{y} \leq y(t) \leq (Fu_0)(t),
\]
\[
(TBu_0)(t) \leq \bar{v} \leq v(t) \leq (TBu_0)(t),
\]
\[
(SBu_0)(t) \leq \bar{w} \leq w(t) \leq (SBu_0)(t).
\]

(H\(_2\)) For any \( r > 0 \), there exist \( d_r \geq 0, d_r^* \geq 0, \) and \( b^{(k)} \geq 0, a^{(k)} \geq 0, \) \( (k = 1, 2, \cdots, m) \) such that
\[
\alpha(f(J, U_1, U_2, U_3, U_4, U_5, U_6)) \leq d_r \alpha(U_3) + d_r^* \alpha(U_4),
\]
\[
\forall U_i \subset B_r, (i = 1, 2, 3, 4, 5, 6), \quad \alpha(I_{3k}(V_1, V_2, V_3, V_4)) \leq b^{(k)} \max\{\alpha(V_1), \alpha(V_4)\},
\]
\[
\forall V_3 \subset B_r, \quad (j = 1, 3, 4, \cdots, m), \quad \alpha(I_{2k}(V_4)) \leq a^{(k)} \alpha(V_4), \forall V_4 \subset B_r, \quad (k = 1, 2, \cdots, m),
\]
where \( B_r = \{u \in E \mid \|u\| \leq r\} \). \( \alpha \) is the measure of noncompactness in \( E \) with the Kuratowski property.

Denote
\[
[u_0, v_0] = \{u \in PC^1[J, E] \mid u_0(t) \leq u(t) \leq v_0(t), u_0(t) \leq u(t) \leq v_0(t), \forall t \in J\}.
\]

**Theorem 3.1.** Suppose \( E \) is a real Banach space, \( P \) is a normal cone, \( B \) and \( F \) are bounded operators, and \((H_1) - (H_3)\) hold, assume that (4) or (5) is satisfied. Then there exist monotone sequences
\[
\{u_n\}, \{v_n\} \subset PC^1[J, E] \cap C^2[J', E],
\]
are uniform convergence at
\[
\{u^*, v^*\} \subset PC^1[J, E] \cap C^2[J', E],
\]
where \( u^* \) is a minimal solution and \( v^* \) is a maximal solution of (18) on \([u_0, v_0]\) and \([u_n], [v_n]\) are convergent at \((u^*)', (v^*)'\) respectively, and
\[
u_0(t) \leq u_0(t) \leq \cdots \leq u_n(t) \leq \cdots \leq u'(t) \leq u(t) \quad (21)
\]
\[
u^*(t) \leq v^*(t) \leq \cdots \leq v_n(t) \leq \cdots \leq v'(t) \leq v(t) \quad \forall t \in J.
\]
\[
u_0'(t) \leq u_0'(t) \leq \cdots \leq u_n'(t) \leq \cdots \leq (u^*)'(t) \leq u'(t) \leq v'(t) \leq \cdots \leq v_n'(t) \leq \cdots \leq (v^*)'(t) \leq v'(t) \quad \forall t \in J.
\]

\[
\forall t \in J.
\]

**Proof.** For any \( \eta \in [u_0, v_0] \), we consider the solution of linear impulsive differential equation of type
\[
\begin{align*}
\Delta u(t) & = -M_1(t)u(t) - M_2(t)u'(t) + \sigma(t), \forall t \in J, t \neq t_k, \\
\Delta u(t) & = I_{2k}(\eta(t_k)) + C_1(u(t_k) - \eta(t_k)), \quad \Delta u'(t) = I_{3k}(\eta(t_k)), \quad \Delta u''(t) = I_{3k}(\eta(t_k)) - L_k(u(t_k) - \eta(t_k)), \\
\end{align*}
\]
\[
\forall k = 1, 2, \cdots, m.
\]

Next, we show that \( u \) is a unique solution of IVP (22). Let
\[
f_c(t, u, u') = \sigma(t) - M_1(t)u(t) - M_2(t)u'(t), \quad t \in J.
\]

Firstly, we consider the following linear differential equation:
\[
\begin{align*}
u(t) & = f_c(t, u, u'), \quad t \in J, \\
\end{align*}
\]
\[
u(0) = x^*_0, u'(0) = u_0.
\]

It’s easy to prove that \( u \in C^2[J_0, E] \) is a solution of (24) if and only if \( u \in C^1[J_0, E] \)
\[
\begin{align*}
u(t) & = x^*_0 + t(x^*_0 + \int_0^t (t-s)(\sigma(s) - M_1(s)u(s) - M_2(s)u'(s)))ds, \\
\end{align*}
\]
\[
\quad -M_2(s)u'(s))ds.
\]

Let
\[
(Au)(t) = x^*_0 + t(x^*_0 + \int_0^t (t-s)(\sigma(s) - M_1(s)u(s) - M_2(s)u'(s)))ds.
\]

Then
\[
\begin{align*}
(Au)'(t) & = x^*_0 + \int_0^t (t-s)(\sigma(s) - M_1(s)u(s) - M_2(s)u'(s)))ds, \\
\end{align*}
\]
\[
\quad -M_2(s)u'(s))ds.
\]

For any \( u, v \in C^1[J_0, E] \), by (25) and (26) we have
\[ \| (A_0 u)(t) - (A_0 v)(t) \| \leq \int_0^t (t-s) (M_1^* \| u(s) - v(s) \| + M_2^* \| u'(s) - v'(s) \|) ds \]
\[ \leq \int_0^t (M_1^* \| u(s) - v(s) \| + M_2^* \| u'(s) - v'(s) \|) ds \]
\[ \leq (\tau + 1)(M_1^* + M_2^*) t \| u - v \|_{C^{[\tau J_0], E}}, t \in J_0. \]
\[ \| (A_0 u)'(t) - (A_0 v)'(t) \| \leq \int_0^t (M_1^* \| (A_0 u)(s) - (A_0 v)(s) \| + M_2^* \| (A_0 u)'(s) - (A_0 v)'(s) \|) ds \]
\[ \leq (\tau + 1)(M_1^* + M_2^*) \left( \frac{r^n}{n!} \right) \| u - v \|_{C^{[\tau J_0], E}}, t \in J_0. \]
\[ 1 \leq \frac{r^n}{n!} \left( \frac{r^n}{n!} \right) < 1. \]

So by (29), (30) and the Banach fixed point theorem, then \( A_0^* \) has a unique fixed point \( w_0 \in C^1[J_0, E] \).

It means that \( w_0 \in C^2[J_0, E] \) is a unique solution of the (24) such that
\[
\begin{align*}
w_0'(t) &= f_*(t, w_0(t), w_0'), t \in J_0, \\
w_0(0) &= x_0^*, w_0(0) = x_0^*. 
\end{align*}
\]

In the following we consider
\[
\begin{align*}
u'' &= f_*(t, u, u'), t \in J_1, \\
u(t_1^0) &= I_{21}(\eta(t_1^0)) + C_1(w_0(t_1^0) - \eta(t_1^0)) + w_0(t_1^0), \\
u'(t_1^0) &= I_{31}((B\eta)(t_1^0), (F\eta)(t_1^0), \eta(t_1^0), \eta'(t_1^0)) \\
&= L_1(w_0(0) - \eta(t_1^0)) - L_1(w_0(0) - \eta(t_1^0)) + w_0(0) \\
&+ \int_0^t (t-s)(\sigma(s) - M_1(s)u(s) - M_2(s)u'(s)) ds.
\end{align*}
\]
It is easy to prove that \( u \in PC^1[1, J_1] \cap C^2[1, J_1, E] \) is a solution of (32) if and only if
\[
\begin{align*}
u(t_1^0) &= I_{21}(\eta(t_1^0)) + C_1(w_0(0) - \eta(t_1^0)) \\
&+ w_0(0) + (t-t_1^0) I_{31}((B\eta)(t_1^0), (F\eta)(t_1^0), \eta(t_1^0), \eta'(t_1^0)) \\
&- L_1(w_0(0) - \eta(t_1^0)) + w_0(0) \\
&+ \int_0^t (t-s)(\sigma(s) - M_1(s)u(s) - M_2(s)u'(s)) ds, \quad t \in \overline{J_1}.
\end{align*}
\]

Then for any \( t \in \overline{J_1} \) we have
\[
(A_0 u)'(t) = (I_{31}((B\eta)(t_1^0), (F\eta)(t_1^0), \eta(t_1^0), \eta'(t_1^0)) - L_1(w_0(0) - \eta(t_1^0)) + w_0(0) \\
+ \int_0^t (t-s)(\sigma(s) - M_1(s)u(s) - M_2(s)u'(s)) ds,
\]

Obviously, \( A_* : PC^1[\overline{J_1}, E] \to PC^1[\overline{J_1}, E] \).

For any \( u,v \in PC^1[\overline{J_1}, E] \), using the similar method used in (29) we obtain
\[
\| (A_0^* u) - (A_0^* v) \|_{PC^1[\overline{J_1}, E]} \leq (\tau + 1) \left( M_1^* + M_2^* \left( \frac{r^n}{n!} \right) \right) \| u - v \|_{PC^1[\overline{J_1}, E]} \] (34)

By (30), (34) and the Banach fixed point theorem, \( A_* \) has a unique fixed point \( w_* \in PC^1[\overline{J_1}, E] \).

It means that \( w_* \in PC^1[\overline{J_1}, E] \) is a unique solution to (32) such that
\[
\begin{align*}
\begin{cases}
w_i^* &= f_i(t, w_i, w'_i), t \in J_i, \\
w'_i(t) &= I_{21}(\eta_i(t)) + C_i(w_{i-1}(t) - \eta'_i(t)) + w_{i-1}(t), \\
w'_i(t') &= I_{31}(B\eta(t), (F\eta)(t), \eta(t), \eta'(t)) - L_i(w_{i-1}(t) - \eta(t)) \\
&- L'_i(w_{i-1}(t) - \eta'(t)) + w'_{i-1}(t),
\end{cases}
\end{align*}
\] (35)

Again, we want to prove that linear differential equation for any \(i, (i=1, 2, \ldots, m)\)
\[
\begin{align*}
&\begin{cases}
\eta_i = f_i(t, u_i, u_i', t \in J_i, \\
u(t) = I_{21}(\eta(t)) + C_i(w_{i-1}(t) - \eta'(t)) + w_{i-1}(t), \\
u(t') = I_{31}(B\eta(t), (F\eta)(t), \eta(t), \eta'(t)) - L_i(w_{i-1}(t) - \eta(t)) \\
&- L'_i(w_{i-1}(t) - \eta'(t)) + w'_{i-1}(t),
\end{cases}
\end{align*}
\]
has a unique solution \(w_i \in PC^1[J, E] \cap C^2[(t_i, t_{i+1}), E]\) such that
\[
\begin{align*}
w_i^* &= f_i(t, w_i, w'_i), t \in J_i, \\
w'_i(t) &= I_{21}(\eta_i(t)) + C_i(w_{i-1}(t) - \eta'(t)) + w_{i-1}(t), \\
w'_i(t') &= I_{31}(B\eta(t), (F\eta)(t), \eta(t), \eta'(t)) - L_i(w_{i-1}(t) - \eta(t)) \\
&- L'_i(w_{i-1}(t) - \eta'(t)) + w'_{i-1}(t),
\end{align*}
\] (36)

Let
\[
\begin{align*}
\eta_i(t), t \in J_i, \\
w_i(t), t \in J_i, \\
w_i(t), t \in J_m.
\end{align*}
\] (37)

Combing (31) and (35), (36), (37), we have \(u_i \in PC^1[J, E] \cap C^2[J', E]\) is a unique solution of IVP (22).

Putting \(u_i = \eta_i\), Then
\[
A[u_0, v_0] \rightarrow PC^1[J, E] \cap C^2[J', E].
\]

Next we prove two cases:

Case (1): \(u_0 \leq A(u_0), v_0 \leq (A v_0)', \quad A v_0 \geq v_0, (A v_0)' \geq v_0'. \)

Case (2): if \(\eta_i, \eta'_i \in [u_0, v_0] \) and \(\eta_i \leq \eta'_i, \eta'_i \leq \eta'_i \), then \(A \eta_i \leq A \eta_i, (A \eta_i)' \leq (A \eta_i)' \).

First, consider case (1).

Put \(u_i = A(u_0, p = u_0 - u_i). \) By (22), we have
\[
\begin{align*}
u_i^*(t) &= -M_i(t)u_i(t) - M_2(t)u'_i(t) + M_i(t)u_i(t) \\
&+ M_2(t)u'_i(t) - f(t, (Bu_i)(t), (Fu_i)(t), \\
u_0(t), u_0'(t), (Bu_0)(t), (Fu_0)(t), \forall t \in J, t \neq t_k, \quad &\Delta u_i &= I_{2k}(u_i(t'_k)) + C_i(u'_i(t'_k) - u'_i(t_k)), \\
\Delta u_i' &= I_{3k}((Bu_i)(t_k), (Fu_i)(t_k), u_0(t_k), \\
u_0(t_k), u_0'(t_k) - L_k(u_i(t_k) - u_i(t_k)) \\
&- L'_k(u'_i(t_k) - u'_i(t_k)) (k = 1, 2, \ldots, m), \\
u_i(0) = x^*_i, u'_i(0) = 0, \\
\end{align*}
\]

Moreover, by (H_1) we have
\[
\begin{align*}
p_i(t) &= u_i^*(t) - u_i(t) \\
&\leq -M_i(t)p(t) - M_2(t)p'(t), \forall t \in J, t \neq t_k, \quad &\Delta p &= I_{2k}((Bu_i)(t_k), (Fu_i)(t_k), u_0(t_k), \\
u_0(t_k), u_0'(t_k) - L_k(u_i(t_k) - u_i(t_k)) \\
&- L'_k(u'_i(t_k) - u'_i(t_k)) (k = 1, 2, \ldots, m), \\
p_i(0) &= u_i(0) - u_i(0) = u'_i(0) - x^*_i \\
&\leq u_i(0) - x^*_i = p(0) \leq \theta.
\end{align*}
\] (38)

Hence, by Lemma 2.1 we obtain
\[
p(t) \leq \theta, p'(t) \leq \theta, \quad p_i(t) \leq \theta, \forall t \in J.
\]

This means that
\[
u_i \leq A u_0, u_i' \leq (A u_0)'.
\]

In the same way, \(A v_0 \leq v_0, (A v_0)' \leq v_0'. \)

Next, consider case (2):

Let \(\eta_i, \eta'_i \in [u_0, v_0] \) such that \(\eta_i \leq \eta_i, \eta'_i \leq \eta'_i \) and put \(\lambda_i, \lambda'_i \) where \(\lambda_i = \eta_i, \lambda_i = \eta_i \).

Combing (22) and (H_1), we have
\[
p_i(t) = \lambda_i(t) - \lambda_i(t) - M_i(t)p(t) - M_2(t)p'(t) \\
&- (f(t, (Bu_i)(t), (Fu_i)(t), \eta_i(t), \eta'_i(t)) \\
&+ (TB\eta_i(t))(SB\eta_i(t)) - f(t, (Bu_i)(t), (Fu_i)(t), \eta_i(t), \eta'_i(t)) \\
&+ (TB\eta_i(t))(SB\eta_i(t)) - f(t, (Bu_i)(t), (Fu_i)(t), \eta_i(t), \eta'_i(t)) \\
&+ M_2(t)\eta_i(t) - M_i(t)p(t) - M_2(t)p'(t), \forall t \in J, t \neq t_k, \\
\Delta p_i &= \Delta \lambda_i |_{t_k} - \Delta \lambda_i |_{t_k} = I_{2k}(\eta_i(t_k)) \\
&+ C_k(\lambda_i(t_k) - \eta_i(t_k)) - I_{2k}(\eta'_i(t_k)) \\
&- C_k(\lambda'_i(t_k) - \eta'_i(t_k)) = C_kp'(t_k), \\
\Delta p_i' &= \Delta \lambda_i' |_{t_k} - \Delta \lambda_i' |_{t_k} = \lambda_i |_{t_k} - \lambda_i |_{t_k} \\
&+ \lambda_i'(t_k) - \lambda_i'(t_k) = \lambda_i(t_k) \\
&- \lambda_i(t_k) \\
&+ \lambda_i'(t_k) - \lambda_i'(t_k) = \lambda_i(t_k) \\
&- \lambda_i(t_k).
\]

\[
p_i(0) = \lambda_i(0) - \lambda_i(0) = \lambda_i(0) - \lambda_i(0) = p(0) = 0.
\]
Moreover, by Lemma 2.1 we obtain
\[ p(t) \leq \theta, \quad p'(t) \leq \theta, \quad \forall t \in J, \]
this means that
\[ (A\eta_j)(t) \leq (A\eta_j')(t), \quad (A\eta_j')(t) \leq (A\eta_j')(t). \]
Let
\[ u_n = Au_{n-1}, v_n = Av_{n-1}, \quad (n = 1, 2, \cdots). \tag{39} \]
By Case (1) and Case (2), we have
\[ u_0(t) \leq u_1(t) \leq \cdots \leq u_n(t) \leq \cdots \]
\[ v_n(t) \leq v_{n+1}(t) \leq \cdots \leq v_0(t), \quad \forall t \in J. \]
By normality of \( u_n \) and (40), then \( U, U' \) are both bounded sets in \( PC[J, E] \). For any \( \eta \in [u_0, v_0] \),
combing \((H_1)\) and \((H_2)\), we have
\[ u_n(t) + M_1(t)u_0(t) + M_2(t)u_n(t) \]
\[ \leq f(t, (Bu_{n-1})(t), (Fu_{n-1})(t), u_{n-1}(t), u_n(t), (TBu_{n-1})(t),
    (SBu_{n-1})(t)), \]
\[ (SBu_{n-1})(t)) + M_1(t)u_0(t) + M_2(t)u_n(t) \]
\[ \leq f(t, (B\eta_0)(t), (F\eta_0)(t), \eta(t), \eta'(t), (TB\eta_0)(t),
    (SB\eta_0)(t)) + M_1(t)\eta(t) + M_2(t)\eta'(t) \]
\[ \leq f(t, (B\eta_0)(t), (F\eta_0)(t), \eta(t), \eta'_0(t), (TB\eta_0)(t),
    (SB\eta_0)(t)) + M_1(t)\eta(t) + M_2(t)\eta'(t) \]
\[ \leq \eta_0(t) + M_1(t)\eta(t) + M_2(t)\eta'(t). \tag{41} \]
Moreover, we obtain
\[ \{ f(t, (B\eta_0)(t), (F\eta_0)(t), \eta(t), \eta'(t), (TB\eta_0), (SB\eta_0)) \}
\[ + M_1(t)\eta + M_2(t)\eta' \mid \eta \in [u_0, v_0] \}
\]
is a bounded set. Hence, there exists a constant \( \gamma > 0 \) such that
\[ || f(t, (Bu_{n-1})(t), (Fu_{n-1})(t), u_{n-1}(t), u_n(t), (TBu_{n-1})(t),
    (SBu_{n-1})(t)) \]
\[ - M_1(t)(u_{n-1}(t) - u_{n-1}(t)) \] \[ \leq \gamma, \quad \forall t \in J (n = 1, 2, \cdots), \]
and \( \{ \sigma_n \mid n = 1, 2, \cdots \} \) is a bounded set in \( PC[J, E] \), where
\[ \sigma_n(t) = f(t, (Bu_{n-1})(t), (Fu_{n-1})(t), u_{n-1}(t), u_n(t), (TBu_{n-1})(t),
    (SBu_{n-1})(t)) \]
\[ + M_1(t)u_{n-1}(t) + M_2(t)u_n(t). \]
By the definition of \( u_n(t) \) and (23), we have
\[ u_n(t) = x_2^* + t\alpha^* + \int_0^t (t - s)f(s, (Bu_{n-1})(s),
    (Fu_{n-1})(s), u_{n-1}(s), u_n(s), (TBu_{n-1})(s), \]
\[ (SBu_{n-1})(s)) + M_1(s)u_{n-1}(s) + M_2(s)u_n(s)ds \]
\[ + M_2(s)u_n(s)ds \]
\[ + C_1(u'_n(t_k)) \]
\[ + \sum_{0 \leq t < t_k} (I_{2k}(u'_{n-1}(t_k))) \]
\[ + C_1(u'_n(t_k)) \]
\[ + \sum_{0 \leq t < t_k} (I_{2k}(Bu_{n-1})(t_k),
    (Fu_{n-1})(t_k), (TBu_{n-1})(t_k),
    (SBu_{n-1})(t_k)) \]
\[ - L_k(u_n(t_k) - u_{n-1}(t_k)) \]
\[ - L_k(u'_n(t_k) - u'_{n-1}(t_k)), \quad \forall t \in J (n = 1, 2, \cdots). \]
Then, we have
\[ u_n(t) = x_2^* + \int_0^J f(s, (Bu_{n-1})(s), (Fu_{n-1})(s),
    (TBu_{n-1})(s), (SBu_{n-1})(s))) \]
\[ + M_1(s)u_{n-1}(s) + M_2(s)u_n(s)ds \]
\[ - L_k(u_n(t_k) - u_{n-1}(t_k)) \]
\[ - L_k(u'_n(t_k) - u'_{n-1}(t_k)), \quad \forall t \in J (n = 1, 2, \cdots). \]
By (42), (43) and (44), the functions belonging to \( U, U' \) are equiv-continuity on \( J_k (k = 0, 1, 2, \cdots) \).
So by Lemma 2.4, we have \( \forall t \in J \)
\[ \alpha_{pc}'(U) = \max \left\{ \sup_{t \in J} \alpha(U(t)), \sup_{t \in J} \alpha(U'(t)) \right\}. \]
By \((H_3)\), there exist constants \( d \geq 0, d' \geq 0 \) and
\[ b^{(k)} \geq 0, a^{(k)} \geq 0 \quad (k = 1, 2, \cdots, m) \] such that
\[ \alpha(f(t, (BU)(t), (FU)(t), U(t), U'(t), \]
\[ (TB)(t), (SB)(t))) \]
\[ \leq d\alpha(U(t)) + d'\alpha(U'(t)), \quad \forall t \in J. \]
\[ \alpha(I_{2k}((BU)(t_k), (FU)(t_k), U(t_k), U'(t_k))) \]
\[ \leq b^{(k)} \max \{ \alpha(U(t_k)), \alpha(U'(t_k)) \} \]
\[ (k = 1, 2, \cdots, m). \]
Hence, for any \( t \in J, \) combing (43), (45), (46), (47)
and Lemma 2.3, we have
\[ \alpha(U(t)) \leq 2\alpha \int_0^t \left( \alpha(f(s, (BU)(s), (FU)(s), \]
\[ U(s), U'(s), (TB)(s), (SB)(s))) \]
\[ + 2M_1\alpha(U(s)) + 2M_2\alpha(U'(s)))ds \]
Let \( m(t) = \max \{ \alpha(U(t)), \alpha(U'(t)) \} \).

Because the functions belonging to \( U, U' \) are equi-continuity on \( J_k (k = 1, 2, \cdots, m) \) and \( U, U' \) are bounded, we have \( m(t) \in PC[J, E] \), \( m(t) \geq 0 \). Combing (48) and (49), we obtain

\[
m(t) \leq 2(a + 1)(d + d') + 2M_1' \int_0^t m(s) ds + \sum_{0 < t < t_k} (a^{(k)})' + (a + 1) (b^{(k)})' + 2L_3' \int_0^t m(s) ds
\]

Moreover, by Lemma 2.2, we have \( m(t) \leq 0 \), this means \( m(t) \equiv 0, t \in J \), moreover, \( \alpha(U(t)) \equiv 0, \alpha(U'(t)) \equiv 0, t \in J \).

This yields that \( U \) possesses the relatively compactness in \( PC^1[J, E] \), \( U' \) possesses the relatively compactness in \( PC[J, E] \). Hence, by (40) and the normality of \( P \), \( \{ u_n \} \) is convergent at \( u^* \in PC^1[J, E], \{ u'_n \} \) is convergent at \( (u^*)' \) and

\[
\| u_n - u^* \|_{PC} \rightarrow 0, \| u'_n - (u^*)' \|_{PC} \rightarrow 0
\]

Because \( f \) is continuous, by the definition of \( \sigma_n \) and (51), we have

\[
\| \sigma_n - \sigma^* \|_{PC} \rightarrow 0, (n \rightarrow \infty)
\]

where

\[
\sigma^*(t) = f(t, (Bu^*)'(t), (Fu^*)'(t), u^*(t), (u^*)'(t), (TBu^*)'(t), (SBu^*)'(t))
\]

\[
+ M_1(t)u^*(t) + M_2(t)(u^*)'(t)
\]

By (42), (51), (52) and Lebesgue control convergent theorem, we have

\[
\lim_{n \rightarrow \infty} u_n = u^*(t), \lim_{n \rightarrow \infty} u'_n = (u^*)'(t)
\]

Moreover,

\[
u^*(t) = x_2^* + t x_1^* + \int_0^t (t - s)f(s, (Bu^*)'(s), (Fu^*)'(s), (TBu^*)'(s), (SBu^*)'(s)) ds + \sum_{0 < t < t_k} I_{t_k}((u^*)'(t))
\]

\[
 \sum_{0 < t < t_k} (t - t_k) I_{t_k}((Bu^*)'(t)), (Fu^*)'(t), (u^*(t)), (u^*)'(t), \forall t \in J,
\]

\[
(u^*)'(t) = x_3^* + \int_0^t f(s, (Bu^*)'(s), (Fu^*)'(s), (u^*)'(s), (u^*)'(s)) ds + \sum_{0 < t < t_k} I_{t_k}((u^*)'(t))
\]

\[
u^*(t), (u^*)'(t), \forall t \in J.
\]

It is easy to prove that \( u^* \in PC^1[J, E] \cap C^2[J', E] \) is a solution of IVP (18). In the same way, there exists \( v^* \in PC^1[J, E] \cap C^2[J', E] \) such that

\[
\| v_n - v^* \|_{PC} \rightarrow 0, \| v'_n - (v^*)' \|_{PC} \rightarrow 0.
\]

\( v^* \) is a solution of IVP (18), and by (40), we have

\[
u_n(t) \leq u_n(t) \leq \cdots \leq u_n(t) \leq \cdots \leq u'(t) \leq v'(t) \leq \cdots \leq u'(t) \leq \cdots \leq v'(t) \leq v_0(t), \forall t \in J.
\]

For \( u \in PC^1[J, E] \cap C^2[J', E] \) is any solution of IVP (18) on \( \{ u_n, v_n \} \), then

\[
u_n \leq u_n(t) \leq v_0(t), v_n(t) \leq u_n'(t) \leq v_0(t) \forall t \in J.
\]

Assume that \( u_{n-1} \leq u(t) \leq v_{n-1}(t), u_{n-1}' \leq u'(t) \leq v_{n-1}'(t), \forall t \in J \).

Then \( p(t) = u_n(t) - u(t) \). By (22), (39) and (H_2), we have

\[
p^*(t) = -M_1(t)p(t) - M_2(t)p'(t)
\]

\[
- f(t, (Bu^*)'(t), (Fu^*)'(t), u^*(t), u'(t), (TBu^*)'(t), (SBu^*)'(t))
\]

\[
- f(t, (Bu_{n-1})'(t), (Fu_{n-1})'(t), u_{n-1}(t), (TBu_{n-1})'(t), (SBu_{n-1})'(t))
\]

\[
+ M_1(t)(u(t) - u_{n-1}(t))
\]

\[
+ M_2(t)(u'(t) - u_{n-1}'(t))
\]
There exist $t \in J, t \neq t_k$, $\Delta p |_{t=k} = I_{2k} (u_n'(t_k)) + C_k (u_n(t_k) - u_{n-1}(t_k))$

$$L_k (u_n(t_k) - u_{n-1}(t_k))$$

$\Delta p |_{t=k} = I_{2k} ((B u_n)(t_k), (F u_n)(t_k), u_{n-1}(t_k),$

$u_{n-1}'(t_k) - L_k (u_n(t_k) - u_{n-1}(t_k))$

$\leq -L_k p(t_k) - L_k' p(t_k) (k = 1, 2, \ldots, m),$

$p(0) = p(0) = \theta.$

By Lemma 2.1, we have $p(t) \leq \theta, p'(t) \leq \theta, \forall t \in J.$

Moreover, $u_n(t) \leq u(t), u_n'(t) \leq u'(t), \forall t \in J.$ In the same way, we can show that $u(t) \leq v_n(t), u'(t) \leq v'_n(t), \forall t \in J.$

Hence, we obtain $u_n(t) \leq u(t) \leq v_n(t), u_n'(t) \leq u'(t) \leq v'_n(t), \forall t \in J(n = 0, 1, 2, \ldots).$ (54)

Now if $n \to \infty,$ for any $t \in J,$

$u'(t) \leq u(t) \leq v'(t), (u')'(t) \leq u'(t) \leq (v')'(t).$

By (53), then (22) holds. This ends the proof.

**Theorem 3.2.** Suppose $E$ is a real Banach space, $P$ is a regular cone, and $(H_1), (H_2)$ hold. Assume (4) or (5) is satisfied, then (21) holds.

**Proof.** The proof is similar to the proof of Theorem 3.1, the only difference is that we verify relative compactness of $U, U'$ and the regularity of $P$ by (40) instead of $H_3.$ This ends the proof.

**Corollary 3.1.** If $E$ is a weak sequentially complete Banach space, $P$ is a normal cone, $H_1, H_2$ hold, and (4) or (5) is satisfied, then (21) holds.

**Proof.** If $E$ is a weak sequentially complete Banach space, the normality of $P$ is equivalent to the regularity. Hence, (21) holds by Theorem 3.2. This ends the proof.

**Remark 3.1.** $f$ is relative to operators $B, F.$ To my knowledge, in all papers connected with the second order impulsive integro-differential equation has been not investigated this situation, so IVP (18) is a new problem.

**Remark 3.2.** $B, F$ relative to Theorem 3.1 are bounded and continuous operators, however, $B, F$ relative to Theorem 3.2 are continuous and increasing.

## 4. Some results of the four order impulsive differential equations

Let us list the following assumptions for convenience:

$(G_1)$ There exist $y_0, z_0 \in PC^3 [J, E] \cap C^4 [J', E]$ such that $y_0, z_0 \leq 0, y_0'(t) \leq z_0'(t), y_0''(t) \leq z_0''(t), y_0'''(t) \leq z_0'''(t), y_0''''(t) \leq z_0''''(t), t \in J.$

$(G_2)$ There exist $y_0, z_0 \in PC^3 [J, E] \cap C^4 [J', E]$ such that $y_0, z_0 \leq 0, y_0'(t) \leq z_0'(t), y_0''(t) \leq z_0''(t), y_0'''(t) \leq z_0'''(t), y_0''''(t) \leq z_0''''(t), (T y_0)(t), (S z_0)(t), \forall t \in J, t \neq t_k.$

$(G_3)$ There exist $y_0, z_0 \in PC^3 [J, E] \cap C^4 [J', E]$ such that $y_0, z_0 \leq 0, y_0'(t) \leq z_0'(t), y_0''(t) \leq z_0''(t), y_0'''(t) \leq z_0'''(t), y_0''''(t) \leq z_0''''(t), (T y_0)(t), (S z_0)(t), \forall t \in J, t \neq t_k.$

$(G_4)$ There exist $y_0, z_0 \in PC^3 [J, E] \cap C^4 [J', E]$ such that $y_0, z_0 \leq 0, y_0'(t) \leq z_0'(t), y_0''(t) \leq z_0''(t), y_0'''(t) \leq z_0'''(t), y_0''''(t) \leq z_0''''(t), (T y_0)(t), (S z_0)(t), \forall t \in J, t \neq t_k.$

$(G_5)$ There exist $y_0, z_0 \in PC^3 [J, E] \cap C^4 [J', E]$ such that $y_0, z_0 \leq 0, y_0'(t) \leq z_0'(t), y_0''(t) \leq z_0''(t), y_0'''(t) \leq z_0'''(t), y_0''''(t) \leq z_0''''(t), (T y_0)(t), (S z_0)(t), \forall t \in J, t \neq t_k.$

$(G_6)$ There exist $y_0, z_0 \in PC^3 [J, E] \cap C^4 [J', E]$ such that $y_0, z_0 \leq 0, y_0'(t) \leq z_0'(t), y_0''(t) \leq z_0''(t), y_0'''(t) \leq z_0'''(t), y_0''''(t) \leq z_0''''(t), (T y_0)(t), (S z_0)(t), \forall t \in J, t \neq t_k.$

$(G_7)$ There exist $y_0, z_0 \in PC^3 [J, E] \cap C^4 [J', E]$ such that $y_0, z_0 \leq 0, y_0'(t) \leq z_0'(t), y_0''(t) \leq z_0''(t), y_0'''(t) \leq z_0'''(t), y_0''''(t) \leq z_0''''(t), (T y_0)(t), (S z_0)(t), \forall t \in J, t \neq t_k.$

$(G_8)$ There exist $y_0, z_0 \in PC^3 [J, E] \cap C^4 [J', E]$ such that $y_0, z_0 \leq 0, y_0'(t) \leq z_0'(t), y_0''(t) \leq z_0''(t), y_0'''(t) \leq z_0'''(t), y_0''''(t) \leq z_0''''(t), (T y_0)(t), (S z_0)(t), \forall t \in J, t \neq t_k.$
There exist nonnegative constants such that
\[
0 \leq z_0(t) \leq z_0^*(t), \quad z_0^*(t) - z_0(t) \geq z_0^*(t) - z_0(t) \geq x_2^*,
\]
\[
z_0(0) \geq x_2^*, \quad z_0(0) \geq x_1^*, \quad z_0^*(0) \geq x_2^*,
\]
\[
z_0^*(0) - z_0(0) \geq x_2^* - x_1^*.
\]
\[ (58) \]

\[ (G_2) \] There exist \( M_1(t), M_2(t) \) are bounded with \( M_1 \geq 0, M_2 \geq 0 \) on \( J \) and \( M_1, M_2 \in L^1[0, a]. \]
\( C_3, L_k, L_k^* \) \( (k = 1, 2, \cdots, m) \) are all nonnegative constants such that
\[
f(t, x, y, z, u, v, w) - f(t, x', y', z', u', v', w) \geq -M_1(t) \langle z - z' \rangle - M_2(t) \langle u - u' \rangle, \quad \forall t \in J,
\]
\[
I_{0k}(z) \geq I_{0k}(z), \quad I_{1k}(y, z) \geq I_{1k}(y, z),
\]
\[
I_{2k}(u) - I_{2k}(u) = C_k \langle u - u' \rangle
\]
\[
I_{3k}(x, y, z, u) - I_{3k}(x, y, z, u) \geq -L_k \langle z - z' \rangle - L_k^* \langle u - u' \rangle \quad (k = 1, 2, \cdots, m)
\]
where
\[
y_0(t) \leq z_0(t), \quad y_0'(t) \leq y(t) \leq z_0'(t),
\]
\[
y_0^*(t) \leq z_0^*(t) \quad y_0^*(t) \leq u \leq z_0^*(t),
\]
\[
(Ty_0)(t) \leq y(t) \leq (Tz_0)(t), \quad (Ty_0^*)(t) \leq y(t) \leq (Tz_0^*)(t),
\]
\[
(Sy_0)(t) \leq y(t) \leq (Sz_0)(t), \quad \forall t \in J.
\]
\[ (G_3) \]

Theorem 4.1 Suppose \( E \) is a real Banach space, \( P \) is normal cone, and \( (G_1), (G_2), (G_3), (H_3) \) hold. Assume (4) or (5) is satisfied, then IVP (1) has the maximal and minimal solutions
\[
y_0^*, z_0^* \in PC^1(J, E] \cap C^2[J', E],
\]
on \( [y_0^*, z_0^*] \).

**Proof.** Consider IVP (1). Let \( x^*(t) = u(t), t \in J. \)

Then we have
\[
x^*(t) = u(t), \quad \forall t \in J, t \neq t_k,
\]
\[
u^*(t) = f(t, x(t), x'(t), u(t), u'(t), (Tx)(t), (Sx)(t)), \quad \forall t \in J, t \neq t_k,
\]
\[
\Delta x \big|_{t=t_k} = I_{0k}(u(t_k)),
\]
\[
\Delta x' \big|_{t=t_k} = I_{1k}(x'(t_k), u(t_k)),
\]
\[
\Delta u \big|_{t=t_k} = I_{2k}(u(t_k)),
\]
\[
\Delta u' \big|_{t=t_k} = I_{3k}(x(t_k), x'(t_k), u(t_k), u'(t_k))
\]
\[ (59) \]

For any \( u \in PC[J, E], \) we have
\[
x^*(t) = u(t), \quad \forall t \in J, t \neq t_k,
\]
\[
\Delta x \big|_{t=t_k} = I_{0k}(u(t_k)),
\]
\[
\Delta x' \big|_{t=t_k} = I_{1k}(x'(t_k), u(t_k)),
\]
\[
\Delta u \big|_{t=t_k} = I_{2k}(u(t_k)),
\]
\[
\Delta u' \big|_{t=t_k} = I_{3k}(x(t_k), x'(t_k), u(t_k), u'(t_k))
\]
\[ (60) \]

Obviously, if \( x \in PC^1[J, E] \cap C^2[J', E] \) is a solution of (60) if and only if
\[
x(t) = x_0^* + x_1^* + \int_{t_k}^t (t - s)u(s)ds + \sum_{0 \leq t_k \leq t} I_{0k}(u(t_k))
\]
\[
+ \sum_{0 \leq t_k \leq t} (t - t_k)I_{1k}(x'(t_k), u(t_k)) + \sum_{0 \leq t_k \leq t} I_{2k}(u(t_k)),
\]
\[ (61) \]

and
\[
x'(t) = x_1^* + \int_{0}^t u(s)ds + \sum_{0 \leq t_k \leq t} I_{1k}(x'(t_k), u(t_k)).
\]
\[ (62) \]

Let
\[
x(t) = (Bu)(t), t \in J.
\]
\[ (63) \]

\[ (64) \]

Then define two operators \( B, F \),
\[
B: PC[J, E] \rightarrow PC^1[J, E] \cap C^2[J', E],
\]
\[
\]

Next, we show that

(i) \( B \) is bounded and continuous.

When \( m = 3, 4, \cdots \), for any \( y_1, y_2 \in PC[J, E], \) by (62),(63), we have
\[
\left\| \int_{0}^{t} (t - s) \| y_1(s) - y_2(s) \| ds \right\|
\]
\[ \leq \sum_{0 \leq t_k \leq t} \| I_{0k}(y_1(t_k)) - I_{0k}(y_2(t_k)) \|
\]
\[ + \sum_{0 \leq t_k \leq t} \| I_{1k}(y_1(t_k)) \| \| I_{2k}(u_2(t_k)) \| \| I_{3k}(x^*_1(t_k), u(t_k)) \|
\]

\[ \]
Moreover, for \( \sum_{j k} (Y_j (t_k), Y_k (t_k)) \),
\[
-I_{t_k} ((F Y_2) (t_k), Y_2 (t_k)) \leq \frac{a^2}{2} \parallel y_1 - y_2 \parallel_{PC} + \sum_{k=1}^{m} b_{o_k} \parallel y_1 - y_2 \parallel_{PC}
\]
\[
+ a \sum_{k=1}^{m} a_{l_k} ((F Y_1) (t_k) - (F Y_2) (t_k))
\]
\[
+ a \sum_{k=1}^{m} b_{l_k} \parallel y_1 - y_2 \parallel_{PC}
\]
\[
= \frac{a^2}{2} \parallel y_1 - y_2 \parallel_{PC} + \sum_{k=1}^{m} b_{o_k} \parallel y_1 - y_2 \parallel_{PC}
\]
\[
+ a \sum_{k=1}^{m} a_{l_k} ((F Y_1) (t_k) - (F Y_2) (t_k))
\]
\[
+ a \sum_{k=1}^{m} b_{l_k} \parallel y_1 - y_2 \parallel_{PC}
\]
\[
+ a((a_m + 1) \sum_{k=1}^{m-1} a_{l_k} ((F Y_1) (t_k) - (F Y_2) (t_k))
\]
\[
+ a_m t_{m} \parallel y_1 - y_2 \parallel_{PC} + a_m \sum_{k=1}^{m-1} b_{l_k} \parallel y_1 - y_2 \parallel_{PC}
\]
\[
\leq \frac{a^2}{2} \parallel y_1 - y_2 \parallel_{PC} + \sum_{k=1}^{m} b_{o_k} \parallel y_1 - y_2 \parallel_{PC}
\]
\[
+ a \sum_{k=1}^{m} a_{l_k} ((F Y_1) (t_k) - (F Y_2) (t_k))
\]
\[
+ a_m t_{m} \parallel y_1 - y_2 \parallel_{PC}
\]
\[
+ a((a_m + 1) \sum_{k=1}^{m-1} a_{l_k} ((F Y_1) (t_k) - (F Y_2) (t_k))
\]
\[
+ a_m t_{m} + \sum_{k=1}^{m-1} a_{l_k} t_{m} \sum_{j=k+1}^{m} (a_{j} + 1)) \parallel y_1 - y_2 \parallel_{PC}
\]
\[
\text{Hence,}
\]
\[
\parallel (B Y_1) - (B Y_2) \parallel_{PC} \leq N_1^* \parallel y_1 - y_2 \parallel_{PC},
\]
\[
\text{where}
\]
\[
N_1^* = \frac{a^2}{2} + \sum_{k=1}^{m} b_{o_k} + a \sum_{k=1}^{m} b_{o_k} + a((a_m + 1) \sum_{k=1}^{m-1} b_{o_k})
\]
\[
+ \sum_{k=2}^{m-1} a_{l_k} (\prod_{j=k+1}^{m} (a_{j} + 1)) \sum_{i=1}^{k-1} b_{i_k}
\]
\[
+ a_m t_{m} + \sum_{k=1}^{m-1} a_{l_k} t_{m} \sum_{j=k+1}^{m} (a_{j} + 1))
\]
\[
\text{In the same way,}
\]
\[
\parallel (B Y_1) - (B Y_2) \parallel_{PC} \leq N_2^* \parallel y_1 - y_2 \parallel_{PC},
\]
\[
\text{where}
\]
\[
N_2^* = a \sum_{k=1}^{m} b_{o_k} + (a_{im} + \sum_{k=1}^{m-1} b_{o_k})
\]
\[
+ \sum_{k=2}^{m-1} (a_{l_k} (\prod_{j=k+1}^{m} (a_{j} + 1)) \sum_{i=1}^{k-1} b_{i_k}
\]
\[
+ a_m t_{m} + \sum_{k=1}^{m-1} a_{l_k} t_{m} \sum_{j=k+1}^{m} (a_{j} + 1)).
\]
\[
\text{So}
\]
\[
\parallel Y_1 - Y_2 \parallel_{PC} \leq N^* \parallel y_1 - y_2 \parallel_{PC},
\]
\[
\text{where}
\]
\[
N^* = \max \{N_1^*, N_2^* \}. \text{Hence, } B \text{ is bounded and continuous. When } m = 1, 2, \text{ the proof is similar.}
\]
\[
(\text{ii}) \text{ } B \text{ is increasing.}
\]
\[
\text{For any } y_1, y_2 \in PC[J, E], y_1 \leq y_2, \text{ by } (G_2) \text{ and (61), we have}
\]
\[
(B Y_1) (t) - (B Y_2) (t)
\]
\[
= \int_0^t (t-s)(y_1 (s) - y_2 (s)) ds \leq \theta, t \in J_0.
\]
\[
\text{Then}
\]
\[
(B Y_1) (t) \leq (B Y_2) (t), \forall t \in J_0.
\]
\[
\text{In particular, } (B Y_1) (t_1) \leq (B Y_2) (t_1). \text{ Moreover, for any } t \in J_1, \text{ we have}
\]
\[
\sum_{0 \leq j < t} (I_0 ((y_1 (t_j)) - I_0 ((y_2 (t_j)))
\]
\[
+ \sum_{0 \leq j < t} (t - t_j) (I_{l_k} ((B Y_1) (t_k)), y_1 (t_k)) (B Y_2) (t_k), y_2 (t_k))
\]
\[
- I_{l_k} ((B Y_2) (t_k), y_2 (t_k)) - I_{l_1} ((B Y_2) (t_1), y_1 (t_1))
\]
\[
- I_{l_1} ((B Y_2) (t_1), y_2 (t_1))) \leq \theta.
\]
\[
\text{Then}
\]
\[
(B Y_1) (t) \leq (B Y_2) (t), \forall t \in J_1.
\]
\[
\text{In particular, } (B Y_1) (t_2) \leq (B Y_2) (t_2). \text{ In the same way, we have}
\]
\[
(B Y_1) (t) \leq (B Y_2) (t), \forall t \in J_k,
\]
\[
(B Y_1) (t_{k+1}) \leq (B Y_2) (t_{k+1}) (k = 1, 2, \ldots, m).
\]
\[
\text{Hence,}
\]
\[
(B Y_1) (t) \leq (B Y_2) (t), \forall t \in J,
\]
\[
\text{then } B Y_1 \leq B Y_2. \text{ In the same way, } F \text{ is a bounded continuous operator with increasing. Combing (61) and (62), it is easy to show if}
\]
\[
y \in PC^1[J, E] \cap C^2 [J', E],
\]
then \( By \in PC^3[J,E] \cap C^4[J',E] \), and if
\[
y \in PC^3[J,E] \cap C^4[J',E],
\]

then \( Fy \in PC^3[J,E] \cap C^4[J',E] \).

In the same way, we can show \( B:PC^1[J,E] \cap C^2[J',E] \rightarrow PC^0[J,E] \cap C^1[J',E] \)

They are all bounded continuous operators with increasing. Hence, by (59)-(64), IVP (1) is equivalent to IVP (18).

Obviously, if \( u \in PC^3[J,E] \cap C^2[J',E] \) is a solution of IVP (18), then
\[
x(t) \in PC^3[J,E] \cap C^4[J',E]
\]
is a solution of IVP (1) by (63).

Putting \( u_0(t) = y_0^*(t) \), \( v_0(t) = z_0^*(t) \), \( t \in J \), we have
\[
u_0 \leq v_0.
\]
By \((G_1)\), we obtain
\[
y_0(t) = x_0^* + x_0^*t + \int_0^t (t-s)u_0(s)ds
\]
\[
+ \sum_{0 \leq t_0 < t} I_{0k}(u_0(t_0))
\]
\[
+ \sum_{0 \leq t_0 < t} (t-t_0)I_{1k}(y_0^*(t_0),u_0(t_0)), \forall t \in J,
\]
\[
z_0(t) = x_0^* + x_0^*t + \int_0^t (t-s)v_0(s)ds
\]
\[
+ \sum_{0 \leq t_0 < t} I_{0k}(v_0(t_0))
\]
\[
+ \sum_{0 \leq t_0 < t} (t-t_0)I_{1k}(z_0^*(t_0),v_0(t_0)), \forall t \in J,
\]
\[
y_0'(t) = x_0^* + \int_0^t u_0(s)ds
\]
\[
+ \sum_{0 \leq t_0 < t} I_{1k}(y_0^*(t_0),u_0(t_0)), \forall t \in J,
\]
\[
z_0'(t) = x_0^* + \int_0^t v_0(s)ds
\]
\[
+ \sum_{0 \leq t_0 < t} I_{1k}(z_0^*(t_0),v_0(t_0)), \forall t \in J,
\]
then
\[
y_0(t) = (Bu_0)(t), z_0(t) = (Bv_0)(t),
\]
\[
y_0'(t) = (Fu_0)(t), z_0'(t) = (Fv_0)(t), \forall t \in J,
\]
where \( u_0, v_0 \) satisfy \((H_1)\).

By \((G_2)\), it is easy to know that \((H_2)\) holds. Hence, applying Theorem 3.1, there exist the maximal and minimal solutions \( u^*, v^* \in PC^1[J,E] \cap C^2[J',E] \) of IVP (18) on \([u_0,v_0]\).

Let \( y^* = Bu^*, z^* = Bv^* \). Then
\[
y^*, z^* \in PC^3[J,E] \cap C^4[J',E]
\]
and
\[
y^*(t) = x_0^* + x_0^*t + \int_0^t (t-s)u^*(s)ds
\]
\[
+ \sum_{0 \leq t_0 < t} I_{0k}(u^*(t_0))
\]
\[
+ \sum_{0 \leq t_0 < t} (t-t_0)I_{1k}((y^*)'(t_0),u^*(t_0)), \forall t \in J.
\]

By (74), we have
\[
\left\{(y^*)'(t) = u^*(t), \forall t \in J, t \neq t_k,
\right.
\]
\[
\Delta y^* |_{t_k} = I_{0k}(u^*(t_k)),
\]
\[
\Delta ((y^*)'(t)) |_{t_k} = I_{1k}((y^*)'(t_k),u^*(t_k))(k = 1,2,\cdots,m),
\]
\[
(y^*)(0) = x_0^*, (y^*)'(0) = x_0^*,
\]
If there exist \( u^* \) such that (18) and \( y^* \) such that (75), then \( y^* \) is a solution of IVP (1). In the same way, \( z^* \) is a solution of IVP (1). It is easy to verify \( y^*, z^* \in PC^1[J,E] \cap C^2[J',E] \) are the maximal and minimal solutions of IVP of (1) on \([y_0,z_0]\), respectively. This ends the proof.

**Theorem 4.2.** Suppose \( E \) is a real Banach space, \( P \) is a regular cone, and \((G_1),(G_2),(G_3)\) hold. Assume (4) or (5) is satisfied, then there exist the maximal and minimal solutions \( y^*, z^* \in PC^3[J,E] \cap C^4[J',E] \) of IVP (1) on \([y_0,z_0]\), respectively.

**Proof.** The proof is similar to the proof of Theorem 4.1. If Theorem 3.2 satisfies, then there exist \( u^*, v^* \in PC^1[J,E] \cap C^2[J',E] \) the maximal and minimal solutions of IVP (1) respectively. This ends the proof.

**Corollary 4.1.** If \( E \) is a weak sequentially complete Banach space, \( P \) is a normal cone, \((G_1),(G_2),(G_3)\) hold, and (4) or (5) is satisfied, then IVP (1) has the maximal and minimal solutions \( y^*, z^* \in PC^3[J,E] \cap C^4[J',E] \) on \([y_0,z_0]\).

**Proof.** If \( E \) is a weak sequentially complete Banach space, the normality of \( P \) is equivalent to the regularity of \( P \). Hence the conclusion of Corollary 4.1 holds by Theorem 4.2. This ends the proof.

**Theorem 4.3.** Suppose \( E \) is a real Banach space. \( P \) is a regular cone, and \((G_1),(G_2),(G_3),(H_3)\) hold. Assume (4) or (5) is satisfied. If for any \( z,u \in E \),
\[
f(t,x,y,z,u,v,w) \geq f(t,x,y,z,u,v,w)
\]
\[
\quad \forall x \geq x, y \geq y, z \geq z, u \geq u, v \geq v, w \geq w,
\]
then IVP (1) has the maximal and minimal solutions
\[ y^*, z^* \in PC^3[J, E] \cap C^4[J', E] \text{ on } [y_0, z_0]. \]

**Proof.** Similar to the proof of Theorem 4.1, we consider IVP (1). Let \( x'(t) = u(t), t \in J \), then
\[
x(t) = (Bu)(t), x'(t) = (Fu)(t), t \in J.
\]
Hence, IVP (1) is equivalent to IVP (18). Let
\[
u_0(t) = y_0(t), v_0(t) = z_0(t), t \in J.
\]
Then \( u_0 \leq v_0 \). Combining (77) and \( (G_i') \), for any \( t \in J \), we have
\[
y_0(t) = y_0(0) + y_0'(0)t + \int_0^t (t - s)u_0(s)ds + \sum_{0 < t_i < \tau}(t - t_i)\Delta y_0(t_i),
\]
\[
y_0'(t) = y_0(0) + \int_0^t u_0(s)ds + \sum_{0 < t_i < \tau} \Delta y_0(t_i), \quad \forall \tau \in J,
\]
\[
z_0(t) = z_0(0) + z_0'(0)t + \int_0^t (t - s)v_0(s)ds + \sum_{0 < t_i < \tau} \Delta z_0(t_i),
\]
\[
z_0'(t) = z_0(0) + \int_0^t v_0(s)ds + \sum_{0 < t_i < \tau} \Delta z_0(t_i).
\]
It is easy to verify
\[
y_0(t) \leq (Bu_0)(t), y_0'(t) \leq (Fu_0)(t),
\]
\[
(Bv_0)(t) \leq z_0(t), (Fv_0)(t) \leq z_0'(t), \quad t \in J_0.
\]
In particular,
\[
y_0(t_i) \leq (Bu_0)(t_i), y_0'(t_i) \leq (Fu_0)(t_i),
\]
\[
(Bv_0)(t_i) \leq z_0(t_i), (Fv_0)(t_i) \leq z_0'(t_i).
\]
Moreover, we have for any \( k, (k = 1, 2, \cdots, m) \)
\[
y_0(t) \leq (Bu_0)(t), y_0'(t) \leq (Fu_0)(t),
\]
\[
(Bv_0)(t) \leq z_0(t), (Fv_0)(t) \leq z_0'(t), \quad t \in J_k,
\]
\[
y_0(t_{k+1}) \leq (Bu_0)(t_{k+1}), y_0'(t_{k+1}) \leq (Fu_0)(t_{k+1}),
\]
\[
(Bv_0)(t_{k+1}) \leq z_0(t_{k+1}), (Fv_0)(t_{k+1}) \leq z_0'(t_{k+1}).
\]
So we have
\[
y_0 \leq Bu_0, y_0' \leq Fu_0, Bv_0 \leq z_0, Fv_0 \leq z_0'.
\]
Hence, by \( (G_i') \), we know \( (H_1) \) holds. Similar to the proof of Theorem 4.1, we obtain the conclusion. This ends the proof.

**Theorem 4.4.** Suppose \( E \) is a real Banach space. \( P \) is regular cone, \( (G_i'), (G_1), (G_2), (G_3) \) hold. Assume (4) or (5) is satisfied. If for any \( z, u \in E \), (76) holds, then IVP (1) has the maximal and minimal solutions
\[
y^*, z^* \in PC^3[J, E] \cap C^4[J', E]
\]
on \([y_0, z_0]\).

**Proof.** Similar to Theorem 4.3, it is easy to know \( (H_1) \) holds. Then the rest of the proof is similar to the proof of Theorem 4.3. This ends the proof.

**Corollary 4.2.** If \( E \) is a weak sequentially complete Banach space, \( P \) is a normal cone, \( (G_i'), (G_2), (G_3) \) hold. Assume (4) or (5) is satisfied. If for any \( z, u \in E \), (76) holds, then IVP (1) has the maximal and minimal solutions
\[
y^*, z^* \in PC^3[J, E] \cap C^4[J', E]
\]
on \([y_0, z_0]\).

**Proof.** If \( E \) is a weak sequentially complete Banach space, the normality of \( P \) is equivalent to the regularity of \( P \). Hence, the conclusion of Corollary 4.2 holds by Theorem 4.4. This ends the proof.

**Remark 4.1.** In Theorem 3.2 and Theorem 4.2, Theorem 4.4, the condition \( (H_3) \) is more easy to use and verify.

**5 Application**

**Example 5.1.** Consider the following initial value problem for fourth-order impulsive integro-differential equations:
\[
x^{(4)}_n(t) = \frac{1}{3n}(t^2 + x_{2n}(t)) + \frac{1}{4n} x'_n(t) + \frac{t}{18n} \left(\frac{t^2}{2n} - x_n^*(t) \right) + \frac{t^2}{9n} \left(\frac{t}{n} - x_n^*(t) \right)
\]
\[
+ \frac{1}{6n} (t + \int_0^s e^{-\xi} x_n(\xi) d\xi) d\xi
\]
\[
+ \frac{1}{2n} \int_0^s \frac{1}{1 + t + s} x_{2n}(s) d\xi, \forall 0 \leq t \leq 1, t \neq \frac{1}{2},
\]
\[
\Delta x_n^{i} |_{t = \frac{1}{2}} = \frac{1}{25(n+1)^2} x_n^*(\frac{1}{2}),
\]
\[
\Delta x_n^{i} |_{t = \frac{1}{2}} = \frac{1}{12} x_n^*(\frac{1}{2}) + \frac{1}{10n^2} x_n^*(\frac{1}{2})
\]
\[
\Delta x_n^{i} |_{t = \frac{1}{2}} = \frac{1}{4} x_n^*(\frac{1}{2}),
\]
\[
\Delta x_n^{i} |_{t = \frac{1}{2}} = \frac{1}{2n} x_n^*(\frac{1}{2}) + \frac{1}{15} x_n^*(\frac{1}{2}) - \frac{1}{8n} x_n^*(\frac{1}{2}), \quad n = 1, 2, 3, \cdots
\]
\[
x_n(0) = 0, x'_n(0) = 0, x''_n(0) = 0, x''_n(0) = 0
\]

**Conclusion** IVP (78) has the maximal and minimal solutions belonging to \( C^4 \) on \([0, \frac{1}{2}) \cup (\frac{1}{2}, 1]\) such that
\[\begin{align*}
0 \leq x_n(t) & \leq \begin{cases}
\frac{t^4}{24n}, & t \in [0, \frac{1}{2}], \\
\frac{t^4}{12n}, & t \in (\frac{1}{2}, 1],
\end{cases} \\
0 \leq x_n'(t) & \leq \begin{cases}
\frac{t^3}{6n}, & t \in [0, \frac{1}{2}], \\
\frac{t^3}{3n}, & t \in (\frac{1}{2}, 1],
\end{cases} \\
0 \leq x_n''(t) & \leq \begin{cases}
\frac{t^2}{2n}, & t \in [0, \frac{1}{2}], \\
\frac{t^2}{n}, & t \in (\frac{1}{2}, 1],
\end{cases} \\
0 \leq x_n'''(t) & \leq \begin{cases}
\frac{t}{2n}, & t \in [0, \frac{1}{2}], \\
\frac{t}{n}, & t \in (\frac{1}{2}, 1],
\end{cases}
\end{align*}\]

Proof. Let \( E = c_0 = \{ x = (x_1, x_2, \ldots, x_n, \cdots) : x_n \rightarrow 0 \} \) with the norm \( \| x \| = \sup_n |x_n| \).

Then \( P \) is a normal cone in \( E \), and (78) is a initial value problem in \( E \), where

\[
\begin{align*}
a &= 1, k(t, s) = e^{-ts}, h(t, s) = \frac{1}{1 + t + s}, \\
x_0^* &= x_1^* = x_2^* = \cdots = (0, \ldots, 0), \\
x &= (x_1, x_2, \ldots, x_n, \cdots), y = (y_1, y_2, \cdots, y_n, \cdots), \\
z &= (z_1, z_2, \ldots, z_n, \cdots), u = (u_1, u_2, \ldots, u_n, \cdots), \\
v &= (v_1, v_2, \ldots, v_n, \cdots), w = (w_1, w_2, \ldots, w_n, \cdots), \\
f &= (f_1, f_2, \ldots, f_n, \cdots)
\end{align*}\]

and

\[
f_a(t, x, y, z, u, v, w) = \frac{1}{3n} (t^2 + x_n) + \frac{1}{4n} (t + y_n) + \frac{t}{2n} - z_n + \frac{t^2}{9n} - u_n + \frac{1}{2n} w_n,
\]

\[
m = 1, t_1 = \frac{1}{2}, I_{01} = (I_{011}, I_{012}, \cdots, I_{01n}, \cdots),
\]

\[
I_{11} = (I_{111}, I_{112}, \cdots, I_{11n}, \cdots), I_{21} = (I_{211}, I_{212}, \cdots, I_{21n}, \cdots),
\]

\[
I_{31} = (I_{311}, I_{312}, \cdots, I_{31n}, \cdots),
\]

where

\[
I_{01n}(z) = \frac{1}{25(n+1)^2} z_n^n,
\]

\[
I_{11n}(y, z) = \frac{1}{12} y_n + \frac{1}{10n} z_n^n, I_{21n}(u) = \frac{1}{4} u_n^n,
\]

\[
I_{31n}(x, y, z, u) = \frac{1}{2n} x_n^n + \frac{1}{4n} y_n + \frac{1}{15} z_n^n - \frac{1}{8n} u_n^n.
\]

Let \( J = [0, 1] \), obviously,

\[
f \in C(J \times E \times E \times E \times E \times E \times E),
\]

Let \( y_0(t) = (0, 0, \ldots, 0, \cdots), t \in [0, 1], \)

\[
z_0(t) = \begin{cases}
\frac{t^4}{24}, \frac{t^4}{24n}, \cdots, \frac{t^4}{24}, & t \in [0, \frac{1}{2}], \\
\frac{t^4}{12}, \frac{t^4}{24}, \cdots, \frac{t^4}{12}, & t \in (\frac{1}{2}, 1].
\end{cases}
\]

We have

\[
y_0'(t) = (0, 0, \ldots, 0, \cdots), y_0''(t) = (0, 0, \ldots, 0, \cdots), y_0'''(t) = (0, 0, \ldots, 0, \cdots), t \in [0, 1],
\]

\[
y_0^{(4)}(t) = (0, 0, \ldots, 0, \cdots), t \in [0, 1],
\]

\[
z_0'(t) = \begin{cases}
\frac{t^3}{6}, \frac{t^3}{6n}, \cdots, \frac{t^3}{6}, & t \in [0, \frac{1}{2}], \\
\frac{t^3}{3}, \frac{t^3}{6}, \cdots, \frac{t^3}{3}, & t \in (\frac{1}{2}, 1],
\end{cases}
\]

\[
z_0''(t) = \begin{cases}
\frac{t^2}{2}, \frac{t^2}{2n}, \cdots, \frac{t^2}{2}, & t \in [0, \frac{1}{2}], \\
\frac{t^2}{n}, \frac{t^2}{2n}, \cdots, \frac{t^2}{n}, & t \in (\frac{1}{2}, 1],
\end{cases}
\]

\[
z_0'''(t) = \begin{cases}
\frac{t}{2}, \frac{t}{2n}, \cdots, \frac{t}{2}, & t \in [0, \frac{1}{2}], \\
\frac{1}{n}, \frac{2}{n}, \cdots, \frac{1}{n}, & t \in (\frac{1}{2}, 1],
\end{cases}
\]

Hence, we have \( y_0, z_0 \in PC^3[J, E] \cap C^4[J', E], \)

\[
y_0'(t) \leq z_0'(t), y_0''(t) \leq z_0''(t), y_0'''(t) \leq z_0'''(t), t \in J
\]

and

\[
y_0(0) = z_0(0) = (0, 0, \ldots, 0, \cdots) = x_0^*,
\]

\[
y_0'(0) = z_0'(0) = (0, 0, \ldots, 0, \cdots) = x_1^*,
\]

\[
y_0''(0) = z_0''(0) = (0, 0, \ldots, 0, \cdots) = x_2^*,
\]

\[
y_0'''(0) = z_0'''(0) = (0, 0, \ldots, 0, \cdots) = x_3^*,
\]
\[ f_n(t, y_0(t), y_0'(t), y_0''(t), y_0'''(t), (T_0y_0)(t), (S_0y_0)(t)) \]
\[ = \frac{t^2}{3n} + \frac{t}{4n} + \frac{t^3}{36n^2} + \frac{t^3}{9n} + \frac{t}{6n} \geq 0, \forall t \in [0, 1] \]

when \( 0 \leq t \leq \frac{1}{2} \),
\[ f_n(t, z_0(t), z_0'(t), z_0''(t), z_0'''(t), (T_0z_0)(t), (S_0z_0)(t)) \]
\[ \leq \frac{1}{3n}(t^2 + \frac{t^3}{24n}) + \frac{1}{4n}(t + \frac{t^3}{6n}) \]
\[ + \frac{1}{6n}(t + \int_0^t s^4 \, ds) + \frac{1}{2n} \int_0^t s^4 \, ds \leq \frac{1}{n} \]

when \( \frac{1}{2} < t \leq 1 \),
\[ f_n(t, z_0(t), z_0'(t), z_0''(t), z_0'''(t), (T_0z_0)(t), (S_0z_0)(t)) \]
\[ \leq \frac{1}{3n}(t^2 + \frac{t^3}{24n}) + \frac{1}{4n}(t + \frac{t^3}{6n}) + \frac{1}{18n}(t^2 - \frac{t^3}{n}) \]
\[ + \frac{1}{9}(t - \frac{2t}{n}) + \frac{1}{6n}(t + \int_0^t s^4 \, ds) + \frac{1}{2n} \int_0^t s^4 \, ds \leq \frac{2}{n} \]

\[ \Delta y_0 \bigg|_{t=\frac{1}{2}} = (0, 0, \ldots, 0, \ldots) = I_{01}(y_0^{(2)}(\frac{1}{2})) \]
\[ \Delta y_0' \bigg|_{t=\frac{1}{2}} = (0, 0, \ldots, 0, \ldots) = I_{11}(y_0^{(2)}(\frac{1}{2}), y_0^{(2)}(\frac{1}{2})) \]
\[ \Delta y_0'' \bigg|_{t=\frac{1}{2}} = (0, 0, \ldots, 0, \ldots) = I_{21}(y_0^{(2)}(\frac{1}{2})) \]
\[ \Delta y_0''' \bigg|_{t=\frac{1}{2}} = (0, 0, \ldots, 0, \ldots) \]
\[ = I_{31}(y_0^{(2)}(\frac{1}{2}), y_0^{(2)}(\frac{1}{2}), y_0^{(2)}(\frac{1}{2}), y_0^{(2)}(\frac{1}{2})) \]

so \( (G_2) \) is satisfied. On the other hand, for any \( t \in J \),
\[ y_0(t) \leq \bar{x} \leq x \leq z_0(t), \quad y_0'(t) \leq \bar{y} \leq y \leq z_0'(t), \]
\[ y_0''(t) \leq \bar{z} \leq z \leq z_0''(t), \quad y_0'''(t) \leq \bar{u} \leq u \leq z_0'''(t), \]
\[ T_0y_0(t) \leq \bar{v} \leq v \leq T_0z_0(t), \quad S_0y_0(t) \leq \bar{w} \leq w \leq S_0z_0(t) \]
we have
\[ f(t, x, y, z, u, v, w) = \]
By (80), we have
\[ f_n^{(i)}(t^{(b)}, x^{(b)}, y^{(b)}, z^{(b)}, u^{(b)}, v^{(b)}, w^{(b)}) \]
\[ \leq \frac{1}{3n}(1 + \| x^{(b)} \|) + \frac{1}{4n} \left( 1 + \| y^{(b)} \| \right) + \frac{1}{18n} \left( 1 + \| z^{(b)} \| \right) + \frac{1}{6n} \left( 1 + r \right) + \frac{1}{18n} \left( 1 + r \right) + \frac{1}{2n} \left( 1 + r \right) \]
\[ \left( b, n = 1, 2, 3, \cdots \right). \]
(82)
So
\[ \{ f_n^{(i)}(t^{(b)}, x^{(b)}, y^{(b)}, z^{(b)}, u^{(b)}, v^{(b)}, w^{(b)}) \} \]
is bounded, moreover, we choose subsequence \( \{ b_i \} \) such that
\[ f_n^{(i)}(t^{(b)}, x^{(b)}, y^{(b)}, z^{(b)}, u^{(b)}, v^{(b)}, w^{(b)}) \to \zeta_n, \]
\[ i \to \infty (n = 1, 2, 3, \cdots). \]
(83)
Combining (82) and (83), we have
\[ \left| \zeta_n \right| \leq \frac{1}{3n}(1 + r) + \frac{1}{4n} \left( 1 + r \right) + \frac{1}{18n} \left( 1 + r \right) + \frac{1}{6n} \left( 1 + r \right) \]
\[ \leq \frac{1}{3n}(1 + r) + \frac{1}{4n} \left( 1 + r \right) + \frac{1}{18n} \left( 1 + r \right) + \frac{1}{2n} \left( 1 + r \right), \]
(84)
so
\[ \zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n, \cdots) \in c_0 = E. \]
For any \( \varepsilon > 0 \), by (82) and (84), there exists a positive integer \( n_0 \) such that
\[ \left| f_n^{(1)}(t^{(b)}, x^{(b)}, y^{(b)}, z^{(b)}, u^{(b)}, v^{(b)}, w^{(b)}) \right| < \varepsilon, \]
\[ \left| \zeta_n \right| < \varepsilon, \forall n > n_0, \quad (i = 1, 2, 3, \cdots). \]
(85)
By (83), there exists a positive integer \( i_0 \) such that
\[ \left| f_n^{(1)}(t^{(b)}, x^{(b)}, y^{(b)}, z^{(b)}, u^{(b)}, v^{(b)}, w^{(b)}) - \zeta_n \right| < \varepsilon, \]
\[ \forall i > i_0, \quad (n = 1, 2, \cdots, n_0). \]
(86)
Then, combining (85) and (86), we have
\[ \left\| f_n^{(1)}(t^{(b)}, x^{(b)}, y^{(b)}, z^{(b)}, u^{(b)}, v^{(b)}, w^{(b)}) - \zeta_n \right\| \]
\[ = \sup_n \left\| f_n^{(1)}(t^{(b)}, x^{(b)}, y^{(b)}, z^{(b)}, u^{(b)}, v^{(b)}, w^{(b)}) - \zeta_n \right\| \]
\[ \leq 2\varepsilon, \forall i > i_0. \]
Hence,
\[ \left\| f_n^{(1)}(t^{(b)}, x^{(b)}, y^{(b)}, z^{(b)}, u^{(b)}, v^{(b)}, w^{(b)}) - \zeta_n \right\| \]
\[ \to 0, \quad i \to \infty. \]
Thus,
\[ \alpha(f^{(1)}(J, U_1, U_2, U_3, U_4, U_5, U_6)) = 0, \]
(87)
On the other hand, applying (81),
\[ \alpha(f^{(2)}(J, U_1, U_2, U_3, U_4, U_5, U_6)) \leq \frac{1}{9}\alpha(U_4), \]
\[ \forall U_i \subset B_i (i = 1, 2, 3, 4, 5, 6). \]
(88)
By (87) and (88), we have
\[ \alpha(f^{(1)}(J, U_1, U_2, U_3, U_4, U_5, U_6)) \leq \frac{1}{9}\alpha(U_4), \]
\[ \forall U_i \subset B_i (i = 1, 2, 3, 4, 5, 6). \]
(89)
In the same way,
\[ \alpha(I_3(V_1, V_2, V_3, V_4)) \leq \frac{1}{15}\alpha(V_3), \]
\[ \forall V_j \subset B_j (j = 1, 2, 3, 4), \]
(90)
\[ \alpha(I_2(V_4)) \leq \frac{1}{4}\alpha(V_4), \forall V_4 \subset B_r. \]
(91)
Hence, \((H_1)\) holds, where
\[ d_r = 0, d_r^\alpha = \frac{1}{9}, d_r^{L_1} = \frac{1}{15}, a_r^\alpha = \frac{1}{4}. \]
Finally, it is easy to prove (76) holds. Then, we have the conclusion by Theorem 4.3. This ends the proof.

References:
[8] Y. X. Li, Z. Liu, Monotone iterative technique for addressing impulsive integro-differential equations in...