The 3-rainbow index of graph operations

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Abstract: A tree T, in an edge-colored graph G, is called *a rainbow tree* if no two edges of T are assigned the same color. A k-*rainbow coloring* of G is an edge coloring of G having the property that for every set S of k vertices of G, there exists a rainbow tree T in G such that $S \subseteq V(T)$. The minimum number of colors needed in a k-rainbow coloring of G is the k-*rainbow index of* G, denoted by $rx_k(G)$. Graph operations, both binary and unary, are an interesting subject, which can be used to understand structures of graphs. In this paper, we will study the 3-rainbow index with respect to three important graph product operations (namely Cartesian product, strong product, lexicographic product) and other graph operations. Firstly, let $G_i (i = 1, 2, \cdots, k)$ be connected graphs and G^* be the Cartesian product of G_i. That is to say, $G^* = G_1 \square G_2 \cdots \square G_k$ $(k \geq 2)$. Then we proved that $rx_3(G^*) \leq \sum_{i=1}^{k} rx_3(G_i)$. And we also get the condition when the equality holds. As a corollary, we obtain an upper bound for the 3-rainbow index of strong product graph. Secondly, we discuss the 3-rainbow index of the lexicographic graph $G[H]$ for connected graphs G and H. And the sharp upper bound is given. Finally, we consider some other simple graph operations : the join of two graphs, split a vertex of a graph and subdivide an edge of a graph. The upper bounds of the 3-rainbow index of the three operation graphs are presented, respectively.

Key–Words: 3-rainbow index; Cartesian product; strong product; lexicographic product.

1 Introduction

All graphs considered in this paper are simple, connected and undirected. We follow the terminology and notation of Bondy and Murty [2]. Let G be a nontrivial connected graph of order n on which is defined an edge coloring, where adjacent edges may be the same color. A path P is a *rainbow path* if no two edges of P are colored the same. The graph G is *rainbow connected* if G contains a u-v rainbow path for every pair u, v of distinct vertices of G. If by coloring c the graph G is rainbow connected , the coloring c is called a rainbow coloring of G. The *rainbow connection number* $rc(G)$ of G, introduced by Chartrand et al. in [7], is the minimum number of colors that results in a rainbow connected graph G.

Rainbow connection has an interesting application for the secure transfer of classified information between agencies (cf. [9]). Although the information must be protected for our national security, procedures should be in place that permit access between appropriate parties. This two fold problems can be addressed by assigning information transfer paths between agencies which may have other agencies as intermediaries that require a large enough number of passwords and firewalls which is prohibitive to intruders, yet small enough to manage (that is, enough so that one or more paths between every pair of agencies have no password repeated). Immediately, a question arises: What is the minimum number of passwords or firewalls needed that allows one or more secure paths between every two agencies so that the passwords along each path are distinct? This situation can be modeled by a graph and studied by the means of rainbow coloring.

Later, another generalization of rainbow connection number was introduced by Chartrand et al.[6] in 2009. A tree T is a *rainbow tree* if no two edges of T are colored the same. Let k be a fixed integer with $2 \leq k \leq n$. An edge coloring of G is called a k-*rainbow coloring* if for every set S of k vertices of G, there exists a rainbow tree in G containing the vertices of S. The k-*rainbow index* $rx_k(G)$ of G is the minimum number of colors needed in a k-rainbow coloring of G. It is obvious that $rc(G) = rx_2(G)$. A tree T is called a *concise* tree if T contains S and $T - v$ is not a tree containing S, where v is any vertex of T. In this paper, we suppose the tree containing S be concise. Since if the given tree T is not concise, we can get a concise tree by deleting some vertices from T.

As we know, the diameter is a natural lower

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bound of the rainbow connection number. Similarly, we consider the Steiner diameter in this paper, which is a nice generalization of the concept of diameter. The *Steiner distance* $d(S)$ of a set S of vertices in G is the minimum size of a tree in G containing S. Such a tree is called a *Steiner S-tree* or simply a *Steiner tree*. The *k-Steiner diameter* $sdiam_k(G)$ of G is the maximum Steiner distance of S among all sets S with k vertices in G. The k-Steiner diameter provides a lower bound for the k-rainbow index of G, i.e., $sdiam_k(G) \leq rx_k(G)$. It follows, for every nontrivial connected graph G of order n, that

$$rx_2(G) \leq rx_3(G) \leq \cdots \leq rx_k(G).$$

For general k, Chartrand et al. [6] determined the k-rainbow index of trees and cycles. They obtained the following theorems.

Theorem 1 *[6] Let T be a tree of order $n \geq 3$. For each integer k with $3 \leq k \leq n$,*

$$rx_k(T) = n - 1.$$

Theorem 2 *[6] For integers k and n with $3 \leq k \leq n$,*

$$rx_k(C_n) = \begin{cases} n-2, & \text{if } k = 3 \text{ and } n \geq 4; \\ n-1, & \text{if } k = n = 3 \text{ or } 4 \leq k \leq n. \end{cases}$$

In this paper, we focus our attention on $rx_3(G)$. For 3-rainbow index of a graph, Chartrand et al. [6] derive the exact value for the complete graphs.

Theorem 3 *[6] For any integer $n \geq 3$,*

$$rx_3(K_n) = \begin{cases} 2, & \text{if } 3 \leq n \leq 5; \\ 3, & \text{if } n \geq 6; \end{cases}$$

Chakraborty et al. [4] showed that computing the rainbow connection number of a graph is NP-hard. So it is also NP-hard to compute k-rainbow index of a connected graph. For rainbow connection number $rc(G)$, people aim to give nice upper bounds for this parameter, especially sharp upper bounds, according to some parameters of the graph G [5, 15, 16, 25].

Many researchers have paid more attention to rainbow connection number of some graph products [1, 13, 10, 18, 19]. There is one way to bound the rainbow connection number of a graph product by the rainbow connection number of the operand graphs. Li and Sun [19] adopted the method to study rainbow connection number with respect to Cartesian product and lexicographic product. They got the following conclusions.

Theorem 4 *[19] Let $G^* = G_1 \square G_2 \cdots \square G_k$ ($k \geq 2$), where each G_i is connected, then*

$$rc(G^*) \leq \sum_{i=1}^{k} rc(G_i)$$

Moreover, if $rc(G_i) = diam(G_i)$ for each G_i, then the equality holds.

Theorem 5 *[19] If G and H are two graphs and G is connected, then we have*
1. if H is complete, then

$$rc(G[H]) \leq rc(G).$$

In particular, if $diam(G) = rc(G)$, then $rc(G[H]) = rc(G)$.
2. if H is not complete, then

$$rc(G[H]) \leq rc(G) + 1.$$

In particular, if $diam(G) = rc(G)$, then $diam(G[H]) = 2$ if G is complete and $diam(G[H]) \leq rc(G) + 1$ if G is not complete.

In this paper, we study the 3-rainbow index with respect to three important graph product operations (namely cartesian product, lexicographic product and strong product) and other operations of graphs. Furthermore, we present the class of graphs which obtain the upper bounds.

1.1 Preliminaries

We use $V(G)$, $E(G)$ for the set of vertices and edges of G, respectively. For any subset X of $V(G)$, let $G[X]$ be the subgraph induced by X, and $E[X]$ the edge set of $G[X]$; Similarly, for any subset E' of $E(G)$, let $G[E']$ be the subgraph induced by E'. For any two disjoint subsets X, Y of $V(G)$, we use $G[X, Y]$ to denote the bipartite graph with vertex set $X \cup Y$ and edge set $E[X, Y] = \{uv \in E(G) | u \in X, v \in Y\}$. The *distance* between two vertices u and v in G is the length of a shortest path between them and is denoted by $d_G(u, v)$. The distance between a vertex u and a path P is the shortest distance between u and the vertices in P. Given a graph G, the eccentricity of a vertex, $v \in V(G)$ is given by $ecc(v) = \max\{d_G(v, u) : u \in V(G)\}$. The diameter of G is defined as $diam(G) = \max\{ecc(v) : v \in V(G)\}$. The length of a path is the number of edges in that path. The length of a tree T is the numbers of edges in that tree, denoted by $size(T)$. $G \setminus e$ denotes the graph obtained by deleting an edge e from the graph G but leaving the vertices and the remaining edges intact. $G - v$ denotes the graph obtained by deleting the vertex v together with all the edges incident with v in G.

Definition 6 *(The Cartesian Product) Given two graphs G and H, the Cartesian product of G and H, denoted by* $G\Box H$*, is defined as follows:* $V(G\Box H) = V(G) \times V(H)$*. Two distinct vertices* (g_1, h_1) *and* (g_2, h_2) *of* $G\Box H$ *are adjacent if and only if either* $g_1 = g_2$ *and* $h_1h_2 \in E(H)$ *or* $h_1 = h_2$ *and* $g_1g_2 \in E(G)$.

Definition 7 *(The Lexicographic Product) The Lexicographic Product* $G[H]$ *of graphs G and H has the vertex set* $V(G[H]) = V(G) \times V(H)$*. Two vertices* $(g_1, h_1), (g_2, h_2)$ *are adjacent if* $g_1g_2 \in E(G)$*, or if* $g_1 = g_2$ *and* $h_1h_2 \in E(H)$.

Definition 8 *(The Strong Product) The Strong Product* $G \boxtimes H$ *of graphs G and H is the graph with* $V(G \boxtimes H) = V(G) \times V(H)$*. Two distinct vertices* (g_1, h_1) *and* (g_2, h_2) *of* $G \boxtimes H$ *are adjacent whenever* $g_1 = g_2$ *and* $h_1h_2 \in E(H)$ *or* $h_1 = h_2$ *and* $g_1g_2 \in E(G)$ *or* $g_1g_2 \in E(G)$ *and* $h_1h_2 \in E(H)$.

Clearly, the resultant graph is isomorphic to G (respectively H) if $H = K_1$ (respectively $G = K_1$). Therefore, we suppose $V(G) \geq 2$ and $V(H) \geq 2$ when studying the 3-rainbow index of these three graph products.

Definition 9 *(The union of graphs) The union of two graphs, by starting with a disjoint union of two graphs G and H and adding edges joining every vertex of G to every vertex of H, the resultant graph is the* join *of G and H, denoted by* $G \vee H$.

Definition 10 *(To split a vertex) To* split *a vertex* v *of a graph G is to replace* v *by two adjacent vertices* v_1 *and* v_2*, and to replace each edge incident to* v *by an edge incident to either* v_1 *or* v_2 *(but not both), the other end of the edge remaining unchanged.*

1.2 Some basic observations

It is easy to see that if the graph H has a 3-rainbow coloring with $rx_3(H)$ colors, then the graph G, which is obtained from H by adding some edges to H, also has a 3-rainbow coloring with $rx_3(H)$ colors since the new edges of G can be colored arbitrarily with the colors used in H. So we have:

Observation 11 *Let G and H be connected graphs and H be a spanning subgraph of G. Then* $rx_3(G) \leq rx_3(H)$.

To verify a 3-rainbow index, we need to find a rainbow tree containing any set of three vertices. So it is necessary to know the structure of concise trees. Next we consider the structure of concise trees T containing three vertices, which will be essential in the sequel.

Observation 12 *Let* G *be a connected graph and* $S = \{v_1, v_2, v_3\} \subseteq V(G)$*. If* T *is a concise tree containing* S*, then* T *belongs to exactly one of Type* I *and Type* II*(see Figure 1).*
Type I*:* T *is a path such that one vertex of* S *as its origin, one of* S *as its terminus, other vertex of* S *as its internal vertex.*
Type II*:* T *is a tree obtained from the star* S_3 *by replacing each edge of* S_3 *with a path* P.

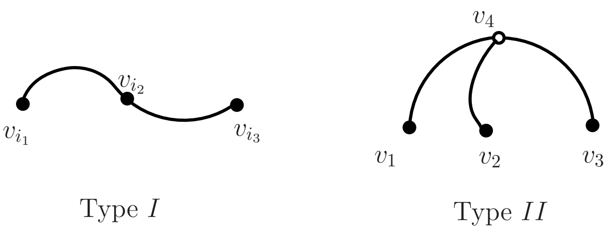

Figure 1: Two types of concise trees, where $\{v_{i_1}, v_{i_2}, v_{i_3}\} = \{v_1, v_2, v_3\}$, $v_4 \in V(G)$

Proof: Firstly, we deduce that the leaves of T belong to S. Since if there exists a leaf v such that $v \notin S$, then we can get the more minimal tree $T' = T - v$ containing S, a contradiction. Thus the T has at most three leaves. If the T has exactly two leaves, then it is easy to verify that T is a path. In this case, T belongs to Type I. Otherwise there is a v_1v_2-path P in T such that $v_3 \notin P$. Since T is connected, there a path P' in T connecting v_3 and P. Let v_4 be the vertex of P' such that $d_T(v_3, v_4)$=$d_T(v_3, P)$. Then we get $T \supseteq P \cup P'$. On the other hand, we know, $P \cup P'$ is a tree containing S. Furthermore, since T is a concise tree, $T = P \cup P'$, which belongs to Type II. □

2 Cartesian product

In this section, we investigate the relationship between the 3-rainbow index of the original graphs and that of the cartesian products. Recall that the Cartesian product of G and H, denoted by $G\Box H$, is defined as follows: $V(G\Box H) = V(G) \times V(H)$. Two distinct vertices (g_1, h_1) and (g_2, h_2) of $G\Box H$ are adjacent if and only if either $g_1 = g_2$ and $h_1h_2 \in E(H)$ or $h_1 = h_2$ and $g_1g_2 \in E(G)$. Let $V(G) = \{g_i\}_{i\in[s]}$, $V(H) = \{h_j\}_{j\in[t]}$. Note that $H_i = G\Box H[\{(g_i, h_j)\}_{j\in[t]}] \cong H, G_j = G\Box H[\{(g_i, h_j)\}_{i\in[s]}] \cong G$. Any edge $(g_i, h_{j_1})(g_i, h_{j_2})$ of H_i corresponds to edge $h_{j_1}h_{j_2}$ of H and $(g_{i_1}, h_j)(g_{i_2}, h_j)$ of G_j corresponds to edge $g_{i_1}g_{i_2}$ of G. For the sake of our results, we give some useful and fundamental conclusions about the Cartesian product.

Lemma 13 *[12] The Cartesian product of two graphs is connected if and only if these two graphs are both connected.*

Lemma 14 *[12] The Cartesian product is associative.*

Lemma 15 *[12] Let (g_1, h_1) and (g_2, h_2) be arbitrary vertices of the Cartesian product $G\Box H$. Then*

$$d_{G\Box H}((g_1, h_1), (g_2, h_2)) = d_G(g_1, g_2) + d_H(h_1, h_2).$$

In the view of Observation 12 and above Lemmas, we derive the following lemma, which is vital to show the sharpness of our main result.

Lemma 16 *Let $G^* = G_1\Box G_2 \cdots \Box G_k$ $(k \geq 2)$, where each G_i is connected. Then*

$$Sdiam_3(G^*) = \sum_{i=1}^{k} Sdiam_3(G_i).$$

Proof: We first prove the conclusion holds for the case $k = 2$. Let $G = G_1$, $H = G_2$, $V(G) = \{g_i\}_{i\in[s]}$, $V(H) = \{h_j\}_{j\in[t]}$, $V(G^*) = \{g_i, h_j\}_{i\in[s],j\in[t]} = \{v_{i,j}\}_{i\in[s],j\in[t]}$. Let $S = \{(g_1, h_1), (g_2, h_2), (g_3, h_3)\}$, $S_1 = \{g_1, g_2, g_3\}$, $S_2 = \{h_1, h_2, h_3\}$ be a set of any three vertices of $V(G^*)$, $V(G)$, $V(H)$, respectively. Suppose that T, T_1 and T_2 be Steiner trees containing S, S_1, S_2, respectively. Next, we only need to show $size(T)$=$size(T_1)$+$size(T_2)$.

On the one hand, by the definition of the Cartesian product of graphs, each edge of G^* is exactly one element of $\{H_i, G_j\}$, $i \in [s]$, $j \in [t]$. Then we can regard T as the union G' and H', where G' is induced by all the edges of $G_j \cap T$, $j \in [t]$, H' is induced by all the edges of $H_i \cap T$, $i \in [s]$. Let G'' and H'' be the graphs induced by the corresponding edges of all edges of $G_j \cap T$ and $H_i \cap T (i \in [s], j \in [t])$ in G and H, respectively. Clearly, G'' and H'' are connected and containing S_1 and S_2, respectively. Hence, we have, $size(T)$=$size(G')$+$size(H')$=$size(G'')$+$size(H'')$ $\geq size(T_1)$+$size(T_2)$.

On the other hand, we try to construct a tree T' containing S with $size(T') = size(T_1)$+$size(T_2)$. Notice that, for every subgraph in G (or H), we can find the corresponding subgraph in any copy G_j (or H_i). If T_1 or T_2 belongs to Type I, without loss of generality, say $T_1 = P_1 \cup P_2$, where P_1 is the path connecting g_{i_1} and g_{i_2}, P_2 is the path connecting g_{i_2} and g_{i_3}, $\{g_{i_1}, g_{i_2}, g_{i_3}\} = \{g_1, g_2, g_3\}$. We can find a tree $T' = P'_1 \cup T'_2 \cup P'_2$ containing S, where the path P'_1 is the corresponding path of P_1 in G_{i_1} and the path P'_2 is the corresponding path of P_2 in G_{i_3}, the tree T'_2 is the

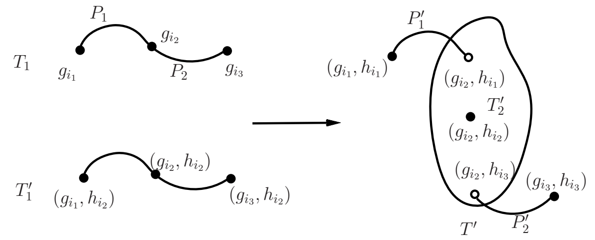

Figure 2 : T_1 belongs to Type I

corresponding tree of T_2 in H_{i_2}, (see Figure 2). If not, that is to say, T_1, T_2 belong to Type II, we suppose $T_1 = P_1 \cup P_2 \cup P_3$, where P_i is the path connecting g_4 and g_i $(1 \leq i \leq 3)$, g_4 is other vertex of G except the vertices of S_1. Then the tree $T' = P'_1 \cup P'_2 \cup P'_3 \cup T'_2$ containing S can also be found in $G\Box H$, where P'_i is the corresponding path of P_i in G_i $(1 \leq i \leq 3)$, the T'_2 is the corresponding tree of T_2 in H_4 (see Figure 3). Thus, $size(T) \leq size(T')$=$size(T_1)$+$size(T_2)$.

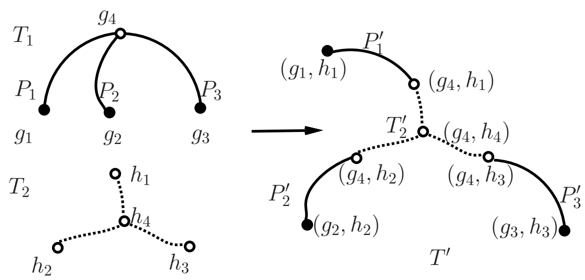

Figure 3 : T_1 and T_2 belong to Type II.

So we get $size(T)$=$size(T_1)$+$size(T_2)$. Hence, $Sdiam_3(G_1\Box G_2)$= $Sdiam_3(G_1)$+$Sdiam_3(G_2)$. By Lemma 14, $Sdiam_3(G^*)$= $Sdiam_3(G_1\Box G_2 \Box \cdots \Box G_{k-1})$+$Sdiam_3(G_k)$=$\sum_{i=1}^{k} Sdiam_3(G_i)$. □

Theorem 17 *Let $G^* = G_1\Box G_2 \cdots \Box G_k$ $(k \geq 2)$, where each G_i is connected, then*

$$rx_3(G^*) \leq \sum_{i=1}^{k} rx_3(G_i)$$

Moreover, if $rx_3(G_i) = Sdiam_3(G_i)$ for each G_i, then the equality holds.

Proof: We first show the conclusion holds for the case $k = 2$. Let $G = G_1$, $H = G_2$, $V(G) = \{g_i\}_{i\in[s]}$, $V(H) = \{h_j\}_{j\in[t]}$, $V(G^*) = \{g_i, h_j\}_{i\in[s],j\in[t]} = \{v_{i,j}\}_{i\in[s],j\in[t]}$. Since G and H are connected, G^* is connected by Lemma 13. For example, Figure 4 shows the case for $G = P_4$ and $H = P_3$.

Since for an edge $v_{i_1,j_1}v_{i_2,j_2} \in G^*$, we have $i_1 = i_2$ or $j_1 = j_2$; if the former, then $v_{i_1,j_1}v_{i_1,j_2} \in H_{i_1}$,

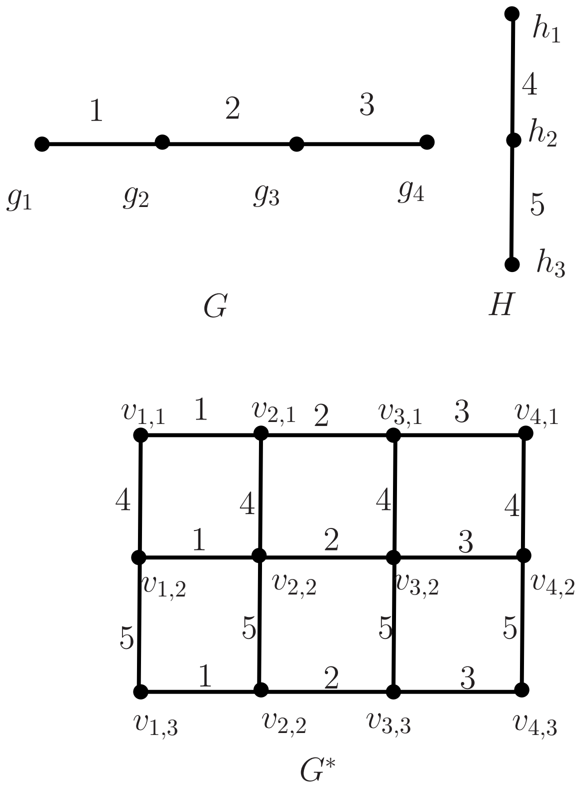

Figure 4 : An example in Theorem 17.

otherwise, $v_{i_1,j_1}v_{i_2,j_1} \in G_{j_1}$. Hence, we only give a coloring of each graph G_j ($j \in [t]$) and H_i ($i \in [s]$).

We give G a 3-rainbow coloring with $rx_3(G)$ colors (see Figure 4 in which G obtains a 3-rainbow coloring with colors 1, 2, 3), and H a 3-rainbow coloring with $rx_3(H)$ fresh colors (see Figure 4 in which H obtains a 3-rainbow coloring with other two fresh colors, 4, 5). Then we color edges of G^* as follow: if the edge belongs to some H_i, then assign the edge with the same color with its corresponding edge of H (for example, edge $v_{1,1}v_{1,2}$ belong to H_1 and corresponds to the edge h_1h_2 in H, so it receives the color 4), otherwise, the edge belongs to some G_j, then assign the edge with the same color with its corresponding edge of G. Now we will show that the given coloring is 3-rainbow coloring of G^*. It suffices to show that for every set S of three vertices of G^*, there is a rainbow tree containing S. Let $S = \{(g_1, h_1), (g_2, h_2), (g_3, h_3)\}$. we distinguish three cases:

Case 1 The vertices of S lie in some G_j (or H_i), where $i, j \in \{1, 2, 3\}$

That is, $g_1 = g_2 = g_3$ or $h_1 = h_2 = h_3$, without loss of generality, we say, $g_1 = g_2 = g_3$. Under the given coloring of H, we can find a rainbow tree T containing h_1, h_2, h_3 in H. By the strategy of the above coloring, the corresponding tree T' of T in H_1 is also rainbow and contains S.

Case 2 The vertices of S lie in two different copies G'_j ,G''_j (or H'_i, H''_i). where j', $j'' \in \{1, 2, 3\}$(or $i', i'' \in \{1 ,2, 3\}$.

Without loss of generality, we assume $g_1 = g_2 \neq g_3$. Note that if a coloring is 3-rainbow coloring, then it is also rainbow coloring, that is, there is a rainbow path connecting any two vertices of graphs. If $h_1 \neq h_2 \neq h_3$ ($h_1 = h_3 \neq h_2$ or $h_2 = h_3 \neq h_1$), we can find a rainbow tree T_1 in H containing h_1, h_2, h_3 (h_1, h_2). By the strategy of coloring, we can find a rainbow tree T'_1 in H_1 containing $\{v_{1,1}, v_{2,2}, v_{1,3},\}$ ($\{v_{1,1}, v_{2,2}\}$). So we can find a rainbow path P'_1 in G_3 connecting $v_{1,3}$ ($v_{1,1}$ or $v_{2,2}$) and $v_{3,3}$. Thus there is a rainbow tree $T = T'_1 \cup P'_1$ in $G\Box H$ containing S.

Case 3 The vertices of S lie in three different copies G_1, G_2, G_3 and H_1, H_2, H_3.

Let T_1 be a rainbow tree containing g_1, g_2, g_3 and T_2 be a rainbow tree containing h_1, h_2, h_3.

If T_1 or T_2 belongs to Type I, say T_1, let $T_1 = P_1 \cup P_2$. Then the tree $T = P'_1 \cup T'_2 \cup P'_2$ containing S can be constructed by the way of Figure 2. And by the character of the given coloring, the tree T is a rainbow tree.

If T_1 and T_2 belong to Type II, let $T_1 = P_1 \cup P_2 \cup P_3$. Then the tree $T = P'_1 \cup P'_2 \cup P'_3 \cup T'_2$ can also be obtained by the way of Figure 3. Furthermore, it is easy to see that the it is also a rainbow tree.

Since we use $rx_3(G) + rx_3(H)$ colors totally, we have $rx_3(G^*) \leq rx_3(G) + rx_3(H)$. From Lemma 16, if $rx_3(G) = Sdiam_3(G)$ and $rx_3(H) = Sdiam_3(H)$, then $Sdiam_3(G^*) = Sdiam_3(G) + Sdiam_3(H) = rx_3(G) + rx_3(H) \geq rx_3(G^*)$. On the other hand, $Sdiam_3(G^*) \leq rx_3(G^*)$, so the conclusion holds for $k = 2$.

For general k, by the Lemma 14, $rx_3(G^*) = rx_3(G_1\Box G_2\Box \cdots \Box G_{k-1}\Box G_k) \leq rx_3(G_1\Box G_2\Box \cdots \Box G_{k-1}) + rx_3(G_k) \leq \sum_{i=1}^{k} rx_3(G_i)$. Moreover, if $rx_3(G) = Sdiam_3(G_i)$ for each G_i, then $rx_3(G^*) \geq Sdiam_3(G^*) = \sum_{i=1}^{k} Sdiam_3(G_i) = \sum_{i=1}^{k} rx_3(G_i) \geq rx_3(G^*)$. So if $rx_3(G_i) = Sdiam_3(G_i)$ for each G_i, then the equality holds. □

Corollary 18 *Let $G = P_{n_1}\Box P_{n_2}\Box \cdots \Box P_{n_k}$, where P_{n_i} is a path with n_i vertices ($1 \leq i \leq k$). Then*

$$rx_3(G) = \sum_{i=1}^{k} n_i - k.$$

Proof: For every path P_{n_i}, by Theorem 1, we have $Sdiam_3(P_{n_i}) = rx_3(P_{n_i}) = n_i - 1$. Thus, by the Theorem 17, $rx_3(G) = \sum_{i=1}^{k} rx_3(P_{n_i}) = \sum_{i=1}^{k} n_i - k$. □

Recall that the strong product $G \boxtimes H$ of graphs G and H has the vertex set $V(G) \times V(H)$. Two vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1 = g_2$ and $h_1h_2 \in E(H)$ or $h_1 = h_2$ and $g_1g_2 \in E(G)$ or $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. By the definition, the graph $G\Box H$ is the spanning subgraph of the graph $G \boxtimes H$ for any graphs G and H. Due to Observation 11, then we have the following result.

Corollary 19 *Let $\overline{G^*} = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_k$, $(k \geq 2)$, where each G_i $(1 \leq i \leq k)$ is connected. Then we have*

$$rx_3(\overline{G^*}) \leq \sum_{i=1}^{k} rx_3(G_i).$$

3 Lexicographic Product

Recall that the lexicographic product $G[H]$ of graphs G and H has the vertex set $V(G[H]) = V(G) \times V(H)$. Two vertices $(g_1, h_1), (g_2, h_2)$ are adjacent if $g_1g_2 \in E(G)$, or if $g_1 = g_2$ and $h_1h_2 \in E(H)$. By definition, $G[H]$ can be obtained from G by submitting a copy H_1 for every $g_1 \in V(G)$ and by joining all vertices of H_1 with all vertices of H_2 if $g_1g_2 \in E(G)$.

In this section, we consider the relationship between 3-rainbow index of the original graphs and their lexicographic product. Since the rainbow connection and 3-rainbow index is only defined in connected graphs, it is nature to assume the original graphs are connected. Note that if $V(G) = 1$ (or $V(H) = 1$), then $G[H]$=H (or G). So in the following discussion, we suppose $V(G) \geq 2$ and $V(H) \geq 2$. By definition, if G and H are complete, then $G[H]$ is also complete.

So for some special cases of G and H, we have the following lemma.

Lemma 20 *If $G, H \cong K_2$, then*

$$rx_3(G[H]) = 2.$$

If G and H are complete with $|V(G)| \geq 3$ or $|V(H)| \geq 3$, then

$$rx_3(G[H]) = 3.$$

Proof: If $G, H \cong K_2$, then $G[H]$=K_4. Hence, we have $rx_3(G[H]) = 2$ by Theorem 3. If G and H are complete with $V(G) \geq 3$ or $V(H) \geq 3$, then $G[H] = K_n$ $(n \geq 6)$. We get immediately $rx_3(G[H]) = 3$ from the Theorem 3. □

For the remaining cases, we obtain the following theorem.

Theorem 21 *Let G and H be two connected graphs with $|V(G)| \geq 2$, $|V(H)| \geq 2$, and at least one of G, H be not complete. Then*

$$rx_3(G[H]) \leq rx_3(G) + rc(H).$$

In particular, if $diam(G) = rx_3(G)$, and H is complete, then the equality holds.

Proof: Let $V(G) = \{g_i\}_{i\in[s]}$, $V(H) = \{h_j\}_{j\in[t]}$, $V(G[H]) = \{g_i, h_j\}_{i\in[s],j\in[t]} = \{v_{i,j}\}_{i\in[s],j\in[t]}$. Let $S = \{(g_1, h_1), (g_2, h_2), (g_3, h_3)\}$ be any three different vertices of $G[H]$. We derive the theorem from two parts: 1. $V(H) = 2$ and G is not complete; 2. $V(H) \geq 3$ and G or H is not complete.

1. If $V(H) = 2$ and G is not complete, we firstly give G a 3-rainbow coloring with $rx_3(G)$ colors. Then we can give $G[H]$ a $rx_3(G)$+1-edge coloring as follows: the edge belongs to some G_j, then assign the edge with the same color with its corresponding edge in G. Otherwise, assign the edge a fresh color.

If $h_1 = h_2 = h_3$, then we can find a rainbow tree T' containing S , which is the corresponding tree of T containing g_1, g_2, g_3 in G_1. Otherwise the vertices of S lie in two different graphs G_1 and G_2. Without loss of generality, we suppose $h_1 = h_3 \neq h_2$. In this case, $(g_1, h_1), (g_3, h_3) \in G_1$, $(g_2, h_2) \in G_2$. Then we can find the corresponding vertex (g_2, h_1) (or (g_1, h_1) or (g_3, h_3)) of (g_2, h_2) in H_1 and a rainbow tree T' containing $(g_1, h_1), (g_3, h_3)$ and (g_2, h_1) (or $\emptyset$). Clearly, there is a rainbow tree $T = T' \cup e$ containing S, where $e = (g_2, h_2)(g_2, h_1)$ (or (g_1, h_1) or (g_3, h_3)). Hence the above coloring is 3-rainbow coloring of $G[H]$. So $rx_3(G[H]) \leq rx_3(G) + 1 = rx_3(G) + rc(H)$.

2. Let $c_1 = \{0, 1, \cdots, rx_3(G) - 1\}$ be a 3-rainbow coloring of G. Let c_2 be a rainbow coloring of H using $rc(H)$ fresh colors. For every $h_j \in H$ color the copy G_j the same as G. By the same way, there is a rainbow tree containing any three vertices $(g_1, h_i), (g_2, h_i), (g_3, h_i) \in V(G[H])$. Every edge of the form $(g_1, h_1)(g_2, h_2)$ get color $k + 1$ mod$(rx_3(G))$, where $g_1g_2 \in E(G)$, $h_1 \neq h_2$, and $c_1(g_1g_2) = k$. Finally, color edges from H_i the same as H such that any two vertices $(g_i, h_j)(g_i, h_k)$ are connected by a rainbow path. The figure 5 shows an example of the coloring.

Now we show the above coloring is 3-rainbow coloring of $G[H]$. We separate into the following three cases.

Case 1 $g_1 = g_2 = g_3$

Since G is a connected graph, there exists an edge $g_1g_4 \in E(G), g_4 \in V(G)$. Then we can find a rainbow path P connecting $(g_2, h_2)(g_1, h_1)$ in H_1, which uses the colors of H. By the coloring of strategy, the

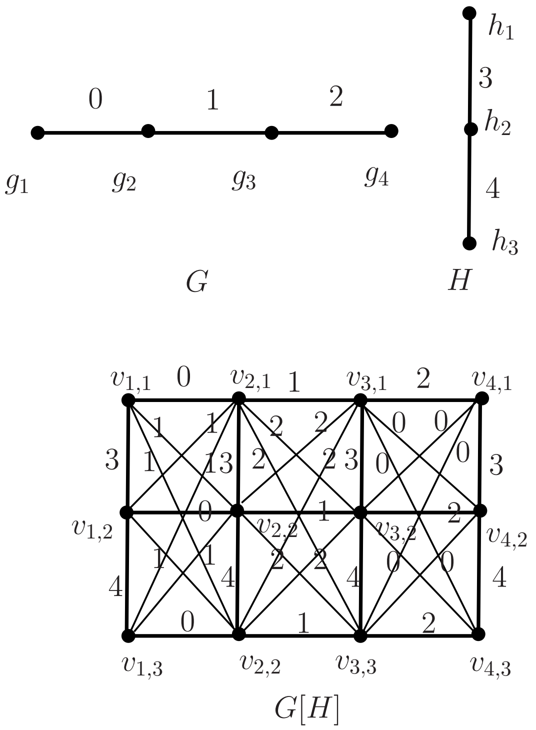

Figure 5 : An example in Theorem 21. 2.

tree $T = P \cup v_{1,1}v_{4,1} \cup v_{4,1}v_{3,3}$ is a rainbow tree containing S.

Case 2 $g_1 = g_2 \neq g_3$ or $g_1 = g_3 \neq g_2$ or $g_2 = g_3 \neq g_1$

Without loss of generality, we assume $g_1 = g_2 \neq g_3$.

Subcase 2.1 $h_1 = h_3$ (or $h_2 = h_3$)

Then $T = P_1 \cup P_2$ is a rainbow tree containing S, where P_1 is a rainbow path connecting (g_1, h_1) and (g_2, h_2) in H_1, P_2 is a rainbow path connecting (g_1, h_1) (or (g_2, h_2)) and (g_3, h_3) in G_3.

Subcases 2.2 $h_1 \neq h_2 \neq h_3$

As we know, there is a rainbow path P_1 connecting g_3 and g_1 in G. The case that P_1=g_3g_1 is trivial, so we assume P_1=$g_3g'_1g'_2, \cdots, g'_kg_1$, $g'_i \in V(G)$ $(1 \leq i \leq k)$. We claim that $P'_1 = (g_3, h_3)(g'_1, h_2)(g'_2, h_3)(g'_3, h_2), \cdots, (g'_k, u)(g_1, h_1)$ is a rainbow path connecting (g_3, h_3) and (g_1, h_1), where $u = h_3$ if k is even and $u = h_2$ otherwise. It is easy to see that the path only use the edge of the form $(g_i, h_j)(g_j, h_l)$, where $g_ig_j \in E(G)$, $h_j \neq h_l$. By the character of coloring, the path is also a rainbow path and only uses the colors of G. Thus, there is a rainbow tree $T = P' \cup P_2$ containing S, where P_2 is a rainbow path connecting (g_1, h_1) and (g_2, h_2) in H_1.

Case 3 $g_1 \neq g_2 \neq g_3$

Subcase 3.1 $h_1 = h_2 = h_3$

Then the S lie in the copy G_1. So by the given coloring, we can claim there is a rainbow tree T containing S.

Subcase 3.2 $h_1 = h_2 \neq h_3$ or $h_1 = h_3 \neq h_2$, or $h_2 = h_3 \neq h_1$

We suppose $h_1 = h_2 \neq h_3$. In this case, we first find the corresponding vertex (g_3, h_1) of (g_3, h_3) in G_1. Then there is a rainbow tree T' containing $(g_1, h_1)(g_2, h_2)(g_3, h_1)$ in G_1 and a rainbow path P connecting $(g_3, h_1)(g_3, h_3)$ in H_3. Thus, the rainbow tree $T = T' \cup P$ is our desire tree.

Subcase 3.3 $h_1 \neq h_2 \neq h_3$

Suppose T_1 be a rainbow tree containing g_1, g_2, g_3.

If T_1 or T_2 belongs to Type I, without loss of generality, we say T_1. In order to describe graphs simply, we might suppose the leaves of T_1 are g_1 and g_3, $T_1 = P_1 \cup P_2$, where P_1 is a rainbow path connecting g_1 and g_2, P_2 is a rainbow path connecting g_2 and g_3. If P_1 or P_2 is an edge, it is trivial. So we suppose $P_1 = g_1g'_1g'_2 \cdots g'_k\ g_2$ and $P_2 = g_2g''_1g''_2 \cdots g''_lg_3$. Thus we can construct a rainbow tree $T'_1 = P'_1 \cup P'_2$ containing S, where $P'_1 = (g_1, h_1)(g'_1, h_3)(g'_2, h_1) \cdots (g'_k, u)(g_2, h_2)$, $P'_2 = (g_2, h_2)(g''_1, h_1)(g''_2, h_2) \cdots (g''_l, v)(g_3, h_3)$, $u = h_3$, if k is odd, $u = h_1$ otherwise; $v = h_1$, if l is odd; $v = h_2$ otherwise.

If T_1 and T_2 belong to Type II, suppose $T_1 = P_1 \cup P_2 \cup P_3$ and $T_2 = Q_1 \cup Q_2 \cup Q_3$, where P_i, Q_i $(1 \leq i \leq 3)$ is a rainbow path connecting g_4 and g_i, h_4 and h_i. If P_i $(1 \leq i \leq 3)$ is an edge, then it is trivial. Now we suppose P_i $(1 \leq i \leq 3)$ are not edges, then P_1=$g_4l'_1l'_2 \cdots l'_kg_1$, $P_2 = g_4l''_1l''_2 \cdots l''_pg_2$, P_3= $g_4l'''_1l'''_2 \cdots l'''_qg_3$. Similarly, the corresponding rainbow tree $T'_1 = P'_1 \cup P'_2 \cup P'_3$ can be obtained containing S, where $P'_1 = (g_4, h_4)(l'_1, h_2)(l'_2, h_4) \cdots (l'_k, u_1)\ (g_1, h_1)$, $P'_2 = (g_4, h_4)(l''_1, h_3)(l''_2, h_4) \cdots (l''_p, u_2)\ (g_2, h_2)$, $P_3 = (g_4, h_4)(l'''_1, h_2)(l'''_2, h_4) \cdots (l'''_q, u_3)(g_3, h_3)$, $u_1, u_3 = h_2$, $u_2 = h_3$ if k, p, q is odd, $u_1, u_2, u_3 = h_4$, otherwise.

From the above discussion, the given coloring is 3-rainbow coloring and we use $rx_3(G)$+$rc(H)$ colors totally. Thus, $rx_3(G[H]) \leq rx_3(G) + rc(H)$.

If $diam(G) = rx_3(G)$, and H is complete, then $rx_3(G[H]) \leq rx_3(G) + rc(H) = diam(G) + 1$. On the other hand, let $g, g' \in V(G)$ such that $d_G(g, g') = diam(G)$. Let $S = \{(g', h), (g, h)(g, h')\}$. By the Lemma 15,it is easy to check that the tree containing S has size at least $diam(G) + 1$. So $rx_3(G[H]) \geq Sdiam_3(G[H]) \geq diam(G)$+1. Thus, $rx_3(G[H]) = rx_3(G) + rc(H)$. □

4 Other graph operations

We first consider the union of two graphs. Recall that the union of two graphs, by starting with a disjoint union of two graphs G and H and adding edges jointing every vertex of G to every vertex of H, the resultant graph is the join of G and H, denoted by $G \vee H$. Note that if $E(G) = \emptyset$ and $E(H) = \emptyset$, then the resultant graph is complete bipartite graph. So we need some results about the 3-rainbow index of complete bipartite graph. Li et al. got the following theorem for regular complete bipartite graphs $K_{r,r}$.

Lemma 22 *[8] For integer r with $r \geq 3$, $rx_3(K_{r,r}) = 3$.*

For complete bipartite graph, we obtained the following Lemmas.

Lemma 23 *[20] For any complete bipartite graphs $K_{s,t}$ with $3 \leq s \leq t$, $rx_3(K_{s,t}) \leq \min\{6, s+t-3\}$, and the bound is tight.*

In the proof of Lemma 23, the claim that for any $s \geq 3$, $t \geq 2 \times 6^s$, $rx_3(K_{s,t}) = 6$ was presented.

Lemma 24 *[21] For any integer $t \geq 2$,*

$$rx_3(K_{2,t}) = \begin{cases} 2, & \text{if } t = 2; \\ 3, & \text{if } t = 3, 4; \\ 4, & \text{if } 5 \leq t \leq 8; \\ 5, & \text{if } 9 \leq t \leq 20; \\ k, & \text{if } C_{k-1}^2 + 1 \leq t \leq C_k^2, (k \geq 6). \end{cases}$$

Then, we derive the relationship between the 3-rainbow index of the original two graphs and that of their join graph. Note that if G and H are both complete graphs, then $G \vee H$ is also the complete graph. By the Theorem 3, $rx_3(G \vee H) = 3$ if $|V(G)|$+$|V(H)| \geq 6$; $rx_3(G \vee H) = 2$ if $|V(G)|$+$|V(H)| \leq 5$. So we consider the remaining cases in following theorem.

Theorem 25 *Let G, H be connected and at least one of G, H be not complete, with $|V(G)| = s$, $|V(H)| = t$, $s \leq t$.*
1. If $s = 1$, then

$$rx_3(G \vee H) \leq rx_3(H) + 1.$$

2. If $2 = s \leq t$, then

$$rx_3(G \vee H) \leq \min\{rc(H) + 3, rx_3(K_{2,t})\}.$$

3. If $3 \leq s \leq t$, then

$$rx_3(G \vee H) \leq \min\{c_1 + 1, rx_3(K_{s,t})\}$$

where $c_1 = \max\{rx_3(G), rx_3(H)\}$.

In particular, if $s = t \geq 3$, then $rx_3(G \vee H) = rx_3(K_{s,t}) = 3$.

Proof: Let $G' = G \vee H$, $V(G') = V_1 \cup V_2$ such that $G'[V_1] \cong G$, $G'[V_2] \cong H$, where $V_1 = \{v_1, v_2, \cdots, v_s\}$, $V_2 = \{u_1, u_2, \cdots, u_t\}$.

1. If $s = 1$, then $G'[V_1]$ is singleton vertex, we give an edge coloring of G' as follows : we first give a 3-rainbow coloring of $G'[V_2]$ using $rx_3(H)$ colors. And for the other edges, that is, elements of $E[V_1, V_2]$, we use a fresh color. It is easy to show the above coloring of G' is 3-rainbow coloring.

2. If $2 = s \leq t$, then $G'[V_1, V_2] \cong K_{2,t}$ is a spanning subgraph of G'. We have $rx_3(G') \leq rx_3(G'[V_1, V_2]) = rx_3(K_{2,t})$. On the other hand, we give an edge coloring of G' as follows: we first color the edges of the subgraph $G'[V_2]$ with $rc(H)$ colors such that it is rainbow connected; we give the elements of $E[V_1, V_2]$ incident with $v_i (i = 1, 2)$ with color $rc(H) + i$ $(i = 1, 2)$; for the element of $G'[V_1]$, we use a fresh color $rc(H) + 3$. It is easy to show the above coloring of G' is 3-rainbow coloring. Thus, we have $rx_3(G \vee H) \leq \min\{rc(H) + 3, rx_3(K_{2,t})\}$.

3. If $3 \leq s \leq t$, by observation 11, we have $rx_3(G') \leq rx_3(G'[V_1, V_2]) = rx_3(K_{s,t})$, similarly. On the other hand, we color the edges of G' as follows: we first color the edges of the subgraph $G'[V_i]$ with c_1 colors such that it is 3-rainbow coloring of $G'[V_i]$ $(i = 1, 2)$. For the rest edges, that is, elements of $E[V_1, V_2]$, we use a fresh color $c_1 + 1$. It is easy to verify that the coloring is a 3-rainbow coloring. Thus, we get $rx_3(G \vee H) \leq \min\{rx_3(K_{s,t}), c_1 + 1\}$.

If $s = t \geq 3$, by Lemma 22, then $rx_3(G') \leq rx_3(K_{s,s}) = 3$; On the other hand, by Observation 11 and Theorem 3, $rx_3(G') \geq rx_3(K_{s+t}) = 3$, so the conclusion holds.

Note that $rx_3(K_{2,t})$ may be larger than $rc(H) + 3$; for example, $H \cong K_t \setminus e$ $(t \geq 21)$. Then $rx_3(K_{2,t}) > 5 = rc(H) + 3$ by Lemma 24. But $rx_3(K_{2,t})$ is not always larger than $rc(H) + 3$; for example, we choose $H \cong P_t$, then $rx_3(K_{2,t}) < t + 2 = rc(H) + 3$. Moreover, $rx_3(K_{s,t})$ $(3 \leq s < t)$ may be larger than $\max\{rx_3(G), rx_3(H)\}+1$, since we suppose $G \cong K_s \setminus e$ $(s \geq 3)$ and $H \cong K_t$, where $t \geq 2 \times 6^s$. Then $rx_3(K_{s,t}) = 6 > \max\{rx_3(G), rx_3(H)\} + 1$. But $rx_3(K_{s,t})$ is not always larger than $\max\{rx_3(G), rx_3(H)\} + 1$. Similarly, for example, $G, H \cong P_s$ $(s \geq 7)$, we can get $\max\{rx_3(G), rx_3(H)\} + 1 = s > 6 \geq rx_3(K_{s,t})$. So the bounds we give in the theorem are reasonable. □

Recall that to split v of a graph G is to replace v by two adjacent vertices v_1 and v_2 by an edge incident to either v_1 or v_2 (but not both), the other end of the edge remaining unchanged. The Figure 6 shows the operation of G. Let $N_G(v)$ be the neighbor sets of v. The set is partitioned into two disjoint sets N_1 and N_2 such that N_1 and N_2 are the neighbor sets of v_1 and

v_2 in the resultant graph, respectively.

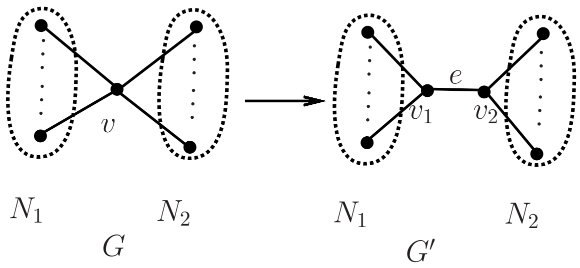

Figure 6 : The operation for vertex spliting.

Theorem 26 *If G is a connected graph and G' is obtained from G by splitting a vertex v, then*

$$rx_3(G') \leq rx_3(G) + 1.$$

Proof: We first give G a 3-rainbow coloring with $rx_3(G)$ colors, then we give G' a $rx_3(G)$+1-edge coloring as follows: we give the edge $e = v_1v_2$ a color $rx_3(G)$+1; for any edge $uv_1 \in G'$ with $uv_1 \neq e$, let the color of uv_1 be the same as that of uv in G; for any edge $v_2w \in G'$ with $v_2w \neq e$, let the color of v_2w be the same as that of vw in G; color of the rest edges of G' are the same as in G. Next, we will show the given coloring of G' is a 3-rainbow coloring. It suffices to show that there is a rainbow tree containing any three vertices of G'. Let $S = \{x, y, z\}$.

Case 1 Two vertices of S belongs to $\{v_1, v_2\}$, say $x = v_1, y = v_2$.

By the above coloring, there a rainbow $v - z$ path $P : v = u_1, \cdots, u_t = z$. If $u_2 \in N_1$, then $P' : v_1, u_2, u_3, \cdots, u_t = z$ is a rainbow connecting z and $x(v_1)$. Thus, $T = P' \cup e$ is the rainbow tree containing S. If $u_2 \in N_2$, it is similar to verify that there is a rainbow tree containing S.

Case 2 Exactly one of S belongs to $\{v_1, v_2\}$, say $x = v_1$.

We know that, in graph G, there is a rainbow tree T_1 containing y, z, v.

subcase 2.1 $d_{T_1}(v) = 1$.

Then there is an edge $uv \in E(T_1)$. If $u \in N_1$, the tree obtained from T_1 by replacing v with v_1 is rainbow and contains S. If $u \in N_2$, the tree obtained from T_1 by replacing v with v_2, v_1 is a rainbow tree containing S.

subcase 2.2 $d_{T_1}(v) \neq 1$.

From the Observation 12, we claim $d_{T_1}(v) = 2$. Let u_1 and u_2 be the two neighbors of v in T_1. If u_1 and u_2 belong to the N_1, then let T be obtained from T_1 by replacing v with v_1. If u_1 and u_2 belong to the N_2, then we can find a rainbow tree $T = T_2 \cup e$, where T_2 is obtained from T_1 by replacing v with v_2. If u_1 and u_2 belong to the different N_i $(i = 1, 2)$, then T obtained from T_1 by replacing v with subgraph v_1v_2 is rainbow.

Case 3 None of vertices in S belongs to $\{v_1, v_2\}$.

We know that there is a rainbow T_3 containing S in G. If v does not belong to T_3, then T_3 is also a rainbow tree containing S in G'.

If v belong to the tree T_3, by the Observation 12, then $d_{T_3}(v) = 2, 3$. Similar to the Subcase 2.2, we can find a rainbow tree containing S.

So G' receives a 3-rainbow coloring. Since we use $rx_3(G) + 1$ colors totally, then $rx_3(G') \leq rx_3(G) + 1$. □

A special case of vertex splitting occurs when exactly one link is assigned to either v_1 or v_2. The resulting graph can be viewed as having been obtained by subdividing an edge of the original graph, where to *subdivide* an edge is to delete e, add a new vertex x, and join x to the ends of e. So by Theorem 26, we have

Corollary 27 *If G is a connected graph, and G' is obtained from G by subdividing an edge e, then*

$$rx_3(G') \leq rx_3(G) + 1.$$

5 Conclusion

We explore the 3-rainbow index of the six graph operations. By constructive proofs, the sharp upper bounds are given. The main results are listed as follows.

(1) Cartesian Product. Let $G^* = G_1 \Box G_2 \cdots \Box G_k$ $(k \geq 2)$, where each $G_i (1 \leq i \leq k)$ is connected. Then $rx_3(G^*) \leq \sum_{i=1}^{k} rx_3(G_i)$ Moreover, if $rx_3(G_i) = Sdiam_3(G_i)$ for each G_i, then the equality holds.

(2) Strong Product. Let $\overline{G^*}$=$G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_k$, $(k \geq 2)$, where each G_i $(1 \leq i \leq k)$ is connected. Then we have $rx_3(\overline{G^*}) \leq \sum_{i=1}^{k} rx_3(G_i)$.

(3) Lexicographic Product. If $G, H \cong K_2$, then $rx_3(G[H]) = 2$; if G and H are complete with $|V(G)| \geq 3$ or $|V(H)| \geq 3$, $rx_3(G[H]) = 3$; if G and H are two connected graphs with $|V(G)| \geq 2$, $|V(H)| \geq 2$, and at least one of G, H is not complete, then $rx_3(G[H]) \leq rx_3(G) + rc(H)$. In particular, if $diam(G) = rx_3(G)$, and H is complete, then the equality holds.

(4) Union of graphs. Let G, H be connected and at least one of G, H be not complete, with $|V(G)| = s$, $|V(H)| = t$, $s \leq t$.

i). If $s = 1$, then $rx_3(G \vee H) \leq rx_3(H) + 1$.

ii). If $2 = s \leq t$, then $rx_3(G \vee H) \leq min\{rc(H) + 3, rx_3(K_{2,t})\}$.

iii). If $3 \leq s \leq t$, then $rx_3(G \vee H) \leq min\{c_1 + 1, rx_3(K_{s,t})\}$ where $c_1 = max\{rx_3(G), rx_3(H)\}$. In particular, if $s = t \geq 3$, then $rx_3(G \vee H) = rx_3(K_{s,t}) = 3$.

(5) Split a vertex. If G is a connected graph and G' is obtained from G by splitting a vertex v, then $rx_3(G') \leq rx_3(G) + 1$.

(6) Subdivide an edge. If G is a connected graph, and G' is obtained from G by subdividing an edge e, then $rx_3(G') \leq rx_3(G) + 1$.

Acknowledgements: The corresponding author, Yumei Hu, is supported by NSFC No. 11001196.

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