On Adaptive Processing of Discrete Flow

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Abstract: An algorithm of approximation for discrete initial numerical flow f by the first order splines, associated with the adaptive grid, is proposed. Quantitative evaluations for volumes of using data are obtained under different characteristics of regularity of initial flow. Limit characteristics of mentioned volumes are represented. Difference representations for deviation of data flow from approximate flow are done. There are discussions for pseudo-equidistant grid, grid of adaptive type and comparison of their effectiveness under condition of the same approximation. The conditions, in which the algorithm is more affective than the analogous algorithm with equidistant grid, are done.

Key–Words: Splines, Interpolation, Approximation, Numerical flow, Adaptive grid, Limit characteristics, Data Volume

1 Introduction

The adaptive methods of numerical solutions of approximation problems are allowed to obtain effective economy of computer resources (run time and storage). They can be applied in BEM/FEM, spline or wavelet approximations. Mentioned methods are connected with choice of adaptive grid. Unless approximated function (initial function) is defined on continuum (see [1, 2]), the selected grid can be varied in mentioned continuum without difficult problems (knots of the grid are freely chosen in the continuum).

Another situation arises if instead of the initial function we have a discrete flow connected with preassigned initial grid (i.e. we have a function defined only on the mentioned grid). We have such situation in different cases of transmission of data with links, of storage of numerical tables, and of procedural representations of usual functions in computer (all such functions are defined only on a computer digital grid, which is nonuniform if floating point is used). In this situation an adaptive grid is an enlargement of the initial grid such that the adaptive grid is subset of the initial grid. Such approach is actual but it imposes the certain restrictions on the construction of adaptive grid.

The aim of the paper is to offer a construction of adaptive grid for approximation of numerical flow; the evaluations are based on the special representation of approximation residual. Comparison of offered method with previous methods shows that the last ones deal with adaptive grids for functions defined on continuum (see below). We discuss more general situation (our functions defined only on initial grid); therefore we need to choose adaptive grid as subset of initial grid. That requires for compliance of additional conditions. The idea of our discussions was blown together by a lot of papers, which we consider below although they deal with functions defined on continuum.

In particular the adaptive grids are used for enlargement of accuracy for solution of problems of mathematical physics (see [3–10]); their application leads up to reduction of size of numerical information flows.

There are a lot of brilliant methods of constructing adaptive grids for approximations of different functions defined on continuum. Some of them are approximations by Taylor series or splines. In this situation it is possible to choose the grid of knots by arbitrary way.

It is known that automatic element mesh generation techniques at this stage have become commonly used tools for the analysis of complex real-world models in 1977 (see [11]).

Automatic methods of triangulations and FEM-approximations for arbitrary differentiable manifolds (and in particular, for smooth surfaces) were represented in 1994 (see [12], pp. 42-72).

The Stretched Grid Method (SGM) allows the obtaining of pseudo-regular meshes very easily and quickly in a one-step solution (see [13]).

An adaptive finite difference scheme for the solution of the discrete first order Hamilton-Jacobi-Bellman equation was presented in the paper [14].

Spherical generalizations of the Delaunay configuration B-spline (DCB-spline) were discussed and adaptive method of new knot-insertion strategies was considered in 2012 (see [15]).

Different adaptive schemes are also discussed in [16], [17], [18] and [19].

The paper [20] was investigated the problem of estimating the support of a structured sparse signal from coordinate-wise observations under the adaptive sensing paradigm.

Adaptive finite element simulation of sheet forming operations using continuum elements is discussed by Ahmed, M. (see [21]).

The paper [22] was presented a greedy search method based on binary successive approximation-evolutionary search (BSA-ES) strategy to design stable infinite impulse response (IIR) digital filter using L1 optimality criterion. A comparison has been made with other design techniques, demonstrating that BSA-ES obtains better results for designing digital IIR filters than the existing genetic algorithm (GA) based methods.

An efficient error indicator with mesh smoothing for mesh refinement were applied to Poisson and Laplace problems (see [23]).

The another approach to treatment of information flows is wavelet decomposition; the last one represent the origin flow as a main flow and an auxiliary flow (wavelet flow) such that it is able to use the main flow instead of the origin flow. The essential property of the decomposition is opportunity to restore the origin flow if it's necessary (see [28]).

Now there are algorithmic basis for construction of spline-wavelets of Lagrange type associated with irregular grids; therefore the union of the both approaches becomes actual thing. Some variants of adaptive grids (with a'priori fixed quantity of used knots) for Lagrange splines are proposed previously (see [29]).

In the offered paper the algorithm of approximation for discrete function u by first order splines associated with the adaptive grid is proposed. The conditions, in which the algorithm is more affective than the analogous algorithm with pseudo-equidistant grid, are done.

2 Auxiliary statements

We discuss the grid Ξ on real interval (α, β),

$$\Xi: \quad \ldots < \xi_{-2} < \xi_{-1} < \xi_0 < \xi_1 < \xi_2 \ldots,$$

$$\lim_{i\to-\infty} \xi_i = \alpha, \quad \lim_{i\to+\infty} \xi_i = \beta.$$

Let $f(t)$ be function defined for $t \in \Xi$, and there is positive c such that

$$f(t) \geq c > 0 \quad \forall t \in \Xi. \tag{1}$$

If $a \in \Xi$, then $a = \xi_i$ for some integer i. By definition, put $a^- \stackrel{\text{def}}{=} \xi_{i-1}$, $a^+ \stackrel{\text{def}}{=} \xi_{i+1}$.

Suppose $a, b \in \Xi$, $a^+ < b^-$. Consider the set $\langle a, b\rangle \stackrel{\text{def}}{=} \{\xi_s \,|\, a \;\leq \xi_s \leq\; b,\ s$ is a integer number $\}$ named by *grid segment*. We discuss a linear space $C\langle a, b\rangle$ that is the set of functions $u(t)$ defined on the grid Ξ:

$$\|u\|_{C\langle a,\, b\rangle} \stackrel{\text{def}}{=} \max_{t\in\langle a,b\rangle} |u(t)|.$$

Assume that

$$\varepsilon \in (\varepsilon^*, \varepsilon^{**}), \tag{2}$$

where

$$\varepsilon^* \stackrel{\text{def}}{=} \max_{\xi\in\langle a,\, b^-\rangle}\ \max_{t\in\{\xi,\xi^+\}} f(t)(\xi^+ - \xi),$$

$$\varepsilon^{**} \stackrel{\text{def}}{=} (b-a)\|f\|_{C\langle a,b\rangle}. \tag{3}$$

Lemma 1. *If conditions (1), (2) – (3) are true, then there exist the unique positive integer $K = K(f, \varepsilon, \Xi)$ and the unique grid*

$$\widetilde{X} = \widetilde{X}(f, \varepsilon, \Xi):$$
$$a = \widetilde{x}_0 < \widetilde{x}_1 < \ldots < \widetilde{x}_K \leq \widetilde{x}_{K+1} = b \tag{4}$$

such that

$$\max_{t\in\langle\widetilde{x}_s,\, \widetilde{x}_{s+1}\rangle} f(t)(\widetilde{x}_{s+1} - \widetilde{x}_s) \leq \varepsilon <$$
$$< \max_{t\in\langle\widetilde{x}_s,\, \widetilde{x}^+_{s+1}\rangle} f(t)(\widetilde{x}^+_{s+1} - \widetilde{x}_s) \tag{5}$$

$$\forall s \in \{0, 1, \ldots, K-1\},$$

$$\max_{t\in\langle\widetilde{x}_K,\, b\rangle} f(t)(b - \widetilde{x}_K) \leq \varepsilon, \quad \widetilde{X} \in X. \tag{6}$$

Proof. Here we use the proof by mathematical induction over the number of knots.

1) The base of mathematical induction is established in this way. Let variable $\tau \in \Xi$ increase from

$a = \widetilde{x}_0$ to b. Taking into account the assumption (2), we see that function $\phi_0(\tau) \stackrel{\text{def}}{=} \max_{t\in\langle\widetilde{x}_0,\,\tau\rangle} f(t)(\tau-\widetilde{x}_0)$ is strictly growing, and if the variable τ increases from $a = \widetilde{x}_0$ to b, then function $\phi_0(\tau)$ increases from 0 to $\max_{t\in\langle a,\,b\rangle} f(t)(b-a)$.

Using the condition (5) – (6), we see that the only one point point $\tau_1 \in \langle a,\ b\rangle$ exists such that

$$\max_{t\in\langle\widetilde{x}_0,\,\tau_1\rangle} f(t)(\tau_1-\widetilde{x}_0) \le \varepsilon < \max_{t\in\langle\widetilde{x}_0,\,\tau_1^+\rangle} f(t)(\tau_1^+-\widetilde{x}_0).$$

Now we put $\widetilde{x}_1 \stackrel{\text{def}}{=} \tau_1$. The base of the mathematical induction has been established.

2). Suppose that knots $\widetilde{x}_1,\ \widetilde{x}_2,\ \ldots,\ \widetilde{x}_s$ of the grid $\widetilde{X}$ have been defined. If $\widetilde{x}_s = b$, then we put $K \stackrel{\text{def}}{=} s - 1$: in that case the construction of the grid $\widetilde{X}(f,\varepsilon,\Xi)$ has been completed. Otherwise we have $\widetilde{x}_s < b$, and the construction of the grid is continued. Consider a function $\phi_s(\tau) \stackrel{\text{def}}{=} \max_{t\in\langle\widetilde{x}_s,\,\tau\rangle} f(t)(\tau-\ \widetilde{x}_s)$; the last one is strictly growing: if τ increases from $\widetilde{x}_s$ to b, then the function $\phi_s(\tau)$ increases from 0 up to $m_s \stackrel{\text{def}}{=} \max_{t\in\langle\widetilde{x}_s,\,b\rangle} f(t)(b-\ \widetilde{x}_s)$. Note that if $\widetilde{x}_s = b^-$, then

$$m_s = \max_{t\in\{b^-,b\}} f(t)(b-b^-) \le$$

$$\le \max_{\xi\in\langle a,b^-\rangle} \max_{t\in\{\xi,\xi^+\}} f(t)(\xi^+-\xi) = \varepsilon^*,$$

and according to supposition (5) – (6), we have $m_s \le\ \varepsilon$. If $m_s \le \varepsilon$, we always put $K \stackrel{\text{def}}{=} s$ and $\widetilde{x}_{s+1} = b$.

Consider the case of $\varepsilon < m_s$; as before we have $\widetilde{x}_s < b^-$. Suppose that the relations $\widetilde{x}_s = \xi_p$ and $m_s = \max_{t\in\langle\widetilde{x}_s,\,\xi_q\rangle} f(t)(\xi_q-\widetilde{x}_s)$ are right for some integer p, q. It's clear that $p < q$ (note that the equality $p = q$ gives $m_s = 0$ but last one contradicts to relation $\varepsilon < m_s$).

Taking into account the inequality $0 < \varepsilon < m_s$, and consider the discrete function $\phi_s(\tau)$ increasing from 0 to m_s, we find j such that $\xi_j \in \langle\widetilde{x}_s,\ b^-\rangle$ and $\phi_s(\xi_j) \le \varepsilon < \phi_s(\xi_{j+1})$; the last one is equivalent to the relation

$$\max_{t\in\langle\widetilde{x}_s,\,\xi_j\rangle} f(t)(\xi_j-\widetilde{x}_s) \le \varepsilon < \max_{t\in\langle\widetilde{x}_s,\,\xi_{j+1}\rangle} f(t)(\xi_{j+1}-\widetilde{x}_s).$$

Now we put $\widetilde{x}_{s+1} \stackrel{\text{def}}{=} \xi_j$.

Existence of the point $\widetilde{x}_{s+1}$, which satisfies to relations (5), is established; the possible meanings of the last one are the next knots of the grid $\xi_{p+1}, \xi_{p+2}, \ldots, \xi_{q-1}$. Uniqueness of the point $\widetilde{x}_{s+1}$ follows from the strict growing of the function $\phi_s(\tau)$.

Thus if $\varepsilon \ge m_s$, the we put $K \stackrel{\text{def}}{=} s$ and $\widetilde{x}_{s+1} = b$; note that the relation (6) is fulfilled. If $\varepsilon < m_s$, then there exists unique point $\tau_{s+1} < b$ such that the inequality (5) holds. Inductive passage has been over.

This concludes the proof.

The net (4) with properties (5) – (6) is called *the net of adaptive type*.

Summing the relations (5), we obtain

$$\sum_{s=0}^{K-1} \max_{t\in\langle\widetilde{x}_s,\,\widetilde{x}_{s+1}\rangle} f(t)(\widetilde{x}_{s+1}-\widetilde{x}_s) \le K\varepsilon <$$

$$< \sum_{s=0}^{K-1} \max_{t\in\langle\widetilde{x}_s,\,\widetilde{x}_{s+1}^+\rangle} f(t)(\widetilde{x}_{s+1}^+-\widetilde{x}_s). \qquad (7)$$

3 On the construction of the net of adaptive type in particular case

Here we give some illustration of the situation in the case of equidistant grid Ξ with knots $\xi_j = jh$. We put

$$a = \xi_0 < \xi_1 < \xi_2 < \xi_0 = b$$

such that $\langle a,\ b\rangle \stackrel{\text{def}}{=} \{0, h, 2h, 3h\}$; thus, $a = 0$, $b = 3h$, and $a^+ = h < b^- = 2h$.

Now we have

$$\varepsilon^{**} = 3\varepsilon^*, \qquad (8)$$

where

$$\varepsilon^* = \max\{f(0), f(h), f(2h), f(3h)\}h = \|f\|_{C\langle a,b\rangle}h.$$

The condition (2) has a form

$$\varepsilon^* < \varepsilon < 3\varepsilon^*. \qquad (9)$$

Let us put

$$\phi_0(\tau) = \max_{t\in\langle 0,\,\tau\rangle} f(t)\tau,$$

where $\tau \in \{0, h, 2h, 3h\}$.

We have

$$\begin{aligned}\phi_0(\tau) &= 0,\\ \phi_0(h) &= h\max\{f(0), f(h)\},\\ \phi_0(3h) &= 2h\max\{f(0), f(h), f(2h)\},\\ \phi_0(3h) &= 3h\|f\|_{C\langle a,b\rangle}.\end{aligned}$$

By definition put $\widetilde{x}_0 \stackrel{\text{def}}{=} a$, $a = 0$. On the first step we find $\widetilde{x}_1$ satisfied to relation (5) with $s = 0$. Thus we find the value τ, $\tau \in \{0, h, 2h\}$, which satisfies to the relation

$$\phi_0(\tau) \leq \varepsilon < \phi_0(\tau^+);$$

the last one may be represented in the form

$$\max_{t\in\langle 0,\, \tau\rangle} f(t)(\tau) \leq \varepsilon < \max_{t\in\langle 0,\, \tau+h\rangle} f(t)(\tau+h).$$

If $\tau = 0$, then we have $0 \leq \varepsilon < \max\{f(0), f(h)\}h$. The last inequality contradicts to the conditions (8) – (9); thus we have only two variants

$$\tau = h \qquad \max\{f(0), f(h)\}h \leq \varepsilon < \\ < 2h \max\{f(0), f(h), f(2h)\}, \quad (10)$$

$$\tau = 2h \qquad 2h \max\{f(0), f(h), f(2h)\} \leq \\ \leq \varepsilon < 3h\|f\|_{C\langle a,b\rangle}, \quad (11)$$

1) If the relation (10) is true, then according to (5) we put

$$\widetilde{x}_1 \stackrel{\text{def}}{=} h, \quad (12)$$

and jump to searching of $\widetilde{x}_2$. For that purpose we consider a function $\phi_1(\tau) = \max\limits_{t\in\langle \widetilde{x}_1,\, \tau\rangle} f(t)(\tau - \widetilde{x}_1)$.

Note that

$$\phi_1(h) = 0, \phi_1(2h) = h \max_{t\in\langle h,\, 2h\rangle} f(t) = \\ = h \max\{f(h), f(2h)\}, \quad (13)$$

$$\phi_1(3h) = 2h \max_{t\in\langle h,\, 3h\rangle} f(t) = \\ = 2h \max\{f(h), f(2h), f(3h)\}. \quad (14)$$

Now we have to find $\tau \in \{h, 2h\}$, which satisfies to relation

$$\phi_1(\tau) \leq \varepsilon < \phi_1(\tau^+). \quad (15)$$

and to put $\widetilde{x}_2 \stackrel{\text{def}}{=} \tau$.

If $\tau = h$, then according to (15) we have

$$0 = \phi_1(h) \leq \varepsilon < \phi_1(2h); \quad (16)$$

it is clear to see that (16) contradicts the conditions (8) – (9).

Now we have the case of $\tau = 2h$; by (15) we have $\phi_1(2h) \leq \varepsilon < \phi_1(3h)$, and (by formulas (13) – (14)) we obtain

$$h \max\{f(h), f(2h)\} \leq \varepsilon < \\ < 2h \max\{f(h), f(2h), f(3h)\}. \quad (17)$$

Thus, if the inequality (17) is fulfilled, then we put $\widetilde{x}_2 \stackrel{\text{def}}{=} 2h$, $\widetilde{x}_3 \stackrel{\text{def}}{=} 3h$. According to the relation (12), we see that the grid $\widetilde{X}$ has been constructed; now we have $\widetilde{X} = \{0, h, 2h, 3h\}$.

If inequality (17) isn't true, then according to (8) – (9), the condition (6) is right; in discussed case the last one has a form

$$2h \max\{f(h), f(2h), f(3h)\} \leq \varepsilon;$$

now we put $\widetilde{x}_2 \stackrel{\text{def}}{=} 3h$. In this case we have $\widetilde{X} = \{0, h, 3h\}$.

2) Up to now we discuss the situation when the inequality (10) is true. Now we suppose that the inequality (11) is realized: $\varphi_0(2h) \leq \varepsilon < \varphi_0(3h)$. In this case we put $\widetilde{x}_1 \stackrel{\text{def}}{=} 2h$ and $\widetilde{x}_2 \stackrel{\text{def}}{=} 3h$; here we have $\widetilde{X} = \{0, 2h, 3h\}$. This concludes the construction of the grid $\widetilde{X}$.

4 Pseudo-equidistant grid

Now suppose that

$$\varepsilon \in (\overline{\varepsilon}^*, \varepsilon^{**}), \quad (18)$$

where

$$\overline{\varepsilon}^* = \max_{\xi\in\langle a,b^-\rangle} (\xi^+ - \xi)\|f\|_{C\langle a,b\rangle}, \varepsilon^{**} = (b-a)\|f\|_{C\langle a,b\rangle}. \quad (19)$$

Then we find the numbers

$$N = N(f, \varepsilon, \Xi) \stackrel{\text{def}}{=} \lfloor \varepsilon^{**}/\varepsilon \rfloor, \quad (20)$$

and

$$h = h(f, \varepsilon, \Xi) \stackrel{\text{def}}{=} \frac{b-a}{N+1}. \quad (21)$$

Consider now the grid $\overline{X} \subset \Xi$,

$$\overline{X} = \overline{X}(f, \varepsilon, \Xi) : \; a = \overline{x}_0 < \overline{x}_1 < \ldots < \overline{x}_N = b, \quad (22)$$

where

$$\overline{x}_{s+1} - \overline{x}_s \leq h < \overline{x}^+_{s+1} - \overline{x}_s, \quad s \in \{0, 1, \ldots, N-1\}, \quad (23)$$

$$\overline{x}_{N+1} - \overline{x}_N \leq h. \quad (24)$$

The grid (22) with properties (23) – (24) is called *pseudo-equidistant grid.*

By (20) we have

$$N \leq \frac{b-a}{\varepsilon}\|f\|_{C\langle a,b\rangle} < N+1; \quad (25)$$

therefore

$$(b-a)\|f\|_{C\langle a,b\rangle} - \varepsilon < N\varepsilon \leq (b-a)\|f\|_{C\langle a,b\rangle}. \quad (26)$$

By (25) we get $\frac{b-a}{N+1}\|f\|_{C\langle a,b\rangle} < \varepsilon$; hence $h\|f\|_{C\langle a,b\rangle} < \varepsilon$. By left inequality (23) and by inequality (24) we obtain

$$\max_{t\in\langle \overline{x}_s,\, \overline{x}_{s+1}\rangle} f(t)\,(\overline{x}_{s+1}-\overline{x}_s) \le \varepsilon, \quad s\in\{0,1,\dots,N\}. \tag{27}$$

By mathematical induction over the number of knots the next assertion can be proved.

Lemma 2. *If the relations (18) – (19) are true, then there exists the unique grid (22) with properties (23) – (24), and the relations (26) – (27) are fulfilled.*

5 Relative quantity of knots and limit relations

In this section we find the boundaries for ratio of the quality of knots for the pseudo-equidistant grid to the quality of knots for the adaptive grid (i.e. the boundaries for N/K).

Theorem 1. *Suppose the hypothesis of lemmas 1 and 2 are fulfilled. Then*

$$\frac{(b-a)\|f\|_{C\langle a,b\rangle}-\varepsilon}{\sum\limits_{s=0}^{K-1}\max\limits_{t\in\langle \widetilde{x}_s,\, \widetilde{x}^+_{s+1}\rangle} f(t)(\widetilde{x}^+_{s+1}-\widetilde{x}_s)} < \frac{N}{K} \le$$

$$\le \frac{(b-a)\|f\|_{C\langle a,b\rangle}}{\sum\limits_{s=0}^{K-1}\max\limits_{t\in\langle \widetilde{x}_s,\, \widetilde{x}_{s+1}\rangle} f(t)(\widetilde{x}_{s+1}-\widetilde{x}_s)}. \tag{28}$$

Proof. In view of suppositions (1) and (4) it's clear that the relation (7) is the inequality between positive expressions.

By (18) – (19) we have $(b-a)\|f\|_{C\langle a,b\rangle}-\varepsilon > 0$; therefore the relation (26) has the same properties: all parts of the relation are positive expressions.

Taking into account these properties, we see that the relation (28) follows from the inequalities (7) and (26).

Note 1. The inequality (28) is easy calculated by computer: instead of search of maximum for function on the segment we find the maximum of finite quantity of given numbers; algorithms of search of the last one are well known (for example, see [30]).

Suppose the function $f(t)$ is continuous in the segment $[a,b]$, and satisfies to the property

$$f(t)\ge c>0 \quad \forall t\in[a,b]. \tag{29}$$

Consider the sequence of grids $\Xi(\lambda)$:

$$\dots<\xi_{-2}(\lambda)<\xi_{-1}(\lambda)<\xi_0(\lambda)<\xi_1(\lambda)<\xi_2(\lambda)\dots \tag{30}$$

depending on parameter $\lambda>0$; suppose $a,b\in\Xi(\lambda)$.

By definition, put

$$\langle a,b\rangle_\lambda \stackrel{\text{def}}{=} \Xi(\lambda)\cap[a,b], \quad h_\lambda \stackrel{\text{def}}{=} \max_{\xi\in\langle a,b^-\rangle_\lambda}(\xi^+-\xi).$$

Theorem 2. *If $f\in C[a,b]$, the condition (5.1) is valid, and*

$$\lim_{\lambda\to+0} h_\lambda = 0, \tag{31}$$

then

$$\lim_{\varepsilon\to+0}\lim_{\lambda\to+0}\frac{K}{N} = \frac{\frac{1}{b-a}\int_a^b f(t)dt}{\|f\|_{C[a,b]}}. \tag{32}$$

Proof. According to (31) it's clear to see that

$$\lim_{\lambda\to+0}\|f\|_{C\langle a,b\rangle_\lambda} = \|f\|_{C[a,b]}.$$

After passage to the limit $\lambda\to+0$ in the relation (7) we can pass to limit under condition $\varepsilon\to+0$; as a result we get the integral $\int_a^b f(t)dt$ instead of sums in the relation (28). The formula (32) is proved.

6 Approximation of the discrete flow

Suppose the function $u(t)$ is defined for $t\in\langle y,z\rangle$, where

$$\langle y,z\rangle: \quad y=\xi_0<\xi_1<\dots<\xi_{M-1}<\xi_M=z. \tag{33}$$

By definition we put

$$\widetilde{u}(t) \stackrel{\text{def}}{=} u(y)+\frac{u(z)-u(y)}{z-y}(t-y), \quad t\in\langle y,z\rangle. \tag{34}$$

Note 2. The formula (34) may be rewritten in the form

$$\widetilde{u}(t) \stackrel{\text{def}}{=} u(z)+\frac{u(z)-u(y)}{z-y}(t-z), \quad t\in\langle y,z\rangle. \tag{35}$$

If $w\in C\langle a,b\rangle$ and

$$y\le\xi_j<\xi_k\le z, \tag{36}$$

then

$$w(\theta)-w(\tau)=\sum_{s=j}^{k-1}\frac{w(\xi_{s+1})-w(\xi_s)}{\xi_{s+1}-\xi_s}(\xi_{s+1}-\xi_s), \tag{37}$$

where $\theta=\xi_k$, $\tau=\xi_j$.

Lemma 3. *If* $w \in C\langle a,b\rangle$, $\theta = \xi_k$, $\tau = \xi_j$, $\xi_k, \xi_j \in\langle a,b\rangle$, then

$$w(\theta)-w(\tau) = \operatorname{sgn}(\theta-\tau) \sum_{s=m(k,j)}^{M(k,j)-1} D_\Xi w(\xi_s)(\xi_{s+1}-\xi_s); \tag{38}$$

here

$$\operatorname{sgn}(q)=\begin{cases} 1 & \text{for } q>0 \\ 0 & \text{for } q=0 \\ -1 & \text{for } q<0, \end{cases} \quad D_\Xi u(\xi)=\frac{u(\xi^+)-u(\xi)}{\xi^+-\xi},$$

$$m(k,j) \stackrel{\text{def}}{=} \min\{k,j\},\ M(k,j) \stackrel{\text{def}}{=} \max\{k,j\}.$$

Proof. Under the condition (36) the relation (38) is equivalent to condition (37); if $y \le \xi_k < \xi_j \le z$, then the relation (38) coincides with the relation (37). If $y \le \xi_k = \xi_j \le z$, then the relation (38) is trivial.

Lemma 4. *If* $t \in \langle y^+, z\rangle$, $t = \xi_k$, *then*

$$u(t) - \widetilde{u}(t) =$$
$$=\sum_{j=0}^{M-1} (\xi_{j+1}-\xi_j)\sum_{i=0}^{k-1}\Big[D_\Xi u(\xi_i)-D_\Xi u(\xi_j)\Big]\frac{\xi_{i+1}-\xi_i}{\xi_M-\xi_0}, \tag{39}$$

$$u(t) - \widetilde{u}(t) =$$
$$=\sum_{j=0}^{M-1} (\xi_{j+1}-\xi_j)\sum_{i=k}^{M-1}\Big[D_\Xi u(\xi_j)-D_\Xi u(\xi_i)\Big]\frac{\xi_{i+1}-\xi_i}{\xi_M-\xi_0}. \tag{40}$$

Proof. Applying (37) to $w = u$, $\tau = y = \xi_0$, $\theta = t = \xi_k$, we have

$$u(t)-u(y) = \sum_{i=0}^{k-1} \frac{u(\xi_{i+1})-u(\xi_i)}{\xi_{i+1}-\xi_i}(\xi_{i+1}-\xi_i). \tag{41}$$

Now we apply the formula (37) with $w = u$, $\tau = y = \xi_0$, $\theta = z = \xi_M$; then

$$u(z)-u(y) = \sum_{j=0}^{M-1} \frac{u(\xi_{j+1})-u(\xi_j)}{\xi_{j+1}-\xi_j}(\xi_{j+1}-\xi_j). \tag{42}$$

Using the formulas (41) – (42), we have

$$u(t) - \widetilde{u}(t) =$$
$$= \frac{1}{\xi_M-\xi_0}\Bigg[(\xi_M-\xi_0)\sum_{i=0}^{k-1} \frac{u(\xi_{i+1})-u(\xi_i)}{\xi_{i+1}-\xi_i}(\xi_{i+1}-\xi_i)-$$
$$-(\xi_k-\xi_0)\sum_{j=0}^{M-1} \frac{u(\xi_{j+1})-u(\xi_j)}{\xi_{j+1}-\xi_j}(\xi_{j+1}-\xi_j)\Bigg]; \tag{43}$$

formulas (43) and (39) are the same.

Consider the difference $u(t)-\widetilde{u}(t)$ again; by (35) we have for $t \in [y,z]$:

$$u(t)-\widetilde{u}(t) = u(t)-u(z) - \frac{u(z)-u(y)}{z-y}(t-z).$$

Applying the formula (38) with $w = u$, $\xi_k = t$, $\xi_j = z$ (i.e. $j = M$) and taking into account $\operatorname{sgn}(t-z) = -1$, $m(k,j) = k$, $M(k,j) = M$, we get

$$u(t)-u(z) = -\sum_{i=k}^{M-1} D_\Xi u(\xi_i)(\xi_{i+1}-\xi_i). \tag{44}$$

Considering the formulas (42) and (44), we obtain

$$u(t)-\widetilde{u}(t) = -\sum_{i=k}^{M-1} D_\Xi u(\xi_i)(\xi_{i+1}-\xi_i)+$$
$$+\frac{\xi_M-\xi_k}{\xi_M-\xi_0}\sum_{j=0}^{M-1} D_\Xi u(\xi_j)(\xi_{j+1}-\xi_j) =$$
$$= \frac{1}{\xi_M-\xi_0}\sum_{j=0}^{M-1}(\xi_{j+1}-\xi_j)\times$$
$$\times\sum_{i=k}^{M-1}(D_\Xi u(\xi_j)-D_\Xi u(\xi_i))(\xi_{i+1}-\xi_i); \tag{45}$$

the relation (40) follows from (45).

Lemma has been proved.

Theorem 3. *If the functions* $u(t)$ *and* $\widetilde{u}(t)$ *are defined on the grid segment* $\langle y,z\rangle$, *then the evaluations*

$$|u(t)-\widetilde{u}(t)| \le 2\min\{t-y, z-t\} \max_{\xi\in\langle y,z^-\rangle} |D_\Xi u(\xi)|, \tag{46}$$

$$|u(t)-\widetilde{u}(t)| \le (z-y) \max_{\xi\in\langle y,z^-\rangle} |D_\Xi u(\xi)|, \quad t\in\langle y,z\rangle \tag{47}$$

are valid.

Proof. If $t = y$ or $t = z$, then left parts of inequalities (46) and (47) are equal to zero, and the right parts of them are nonnegative; therefore the inequalities are obvious in these cases.

If $t \in \langle y^+, z^-\rangle$, then by (39) we find the next expression

$$|u(t)-\widetilde{u}(t)| \le$$
$$\le \frac{1}{\xi_M-\xi_0}\sum_{j=0}^{M-1}(\xi_{j+1}-\xi_j)\sum_{i=0}^{k-1}\big|D_\Xi u(\xi_i)-D_\Xi u(\xi_j)\big|\times$$

$$\times(\xi_{i+1}-\xi_i) \le 2(t-y) \max_{\xi\in\langle y,z^-\rangle} |D_\Xi u(\xi)|; \quad (48)$$

by (40) we have

$$|u(t)-\widetilde{u}(t)| \le 2(z-t) \max_{\xi\in\langle y,z^-\rangle} |D_\Xi u(\xi)|. \quad (49)$$

The relation (46) follows from the formulas (48) and (49), the relation (47) follows from (46).

7 Another variant of approximation for discrete flow

Lemma 5. *If $t \in \langle y^+, z^-\rangle$, then*

$$u(t)-\widetilde{u}(t) = \sum_{j=0}^{M-1}(\xi_{j+1}-\xi_j)\sum_{i=0}^{k-1}(\xi_{i+1}-\xi_i)\times$$

$$\times \mathrm{sgn}(\xi_i-\xi_j)\sum_{p=m(i,j)}^{M(i,j)-1} D^2_\Xi u(\xi_{p+1})(\xi_{p+1}-\xi_p)/(\xi_M-\xi_0), \quad (50)$$

where

$$D^2_\Xi u(\xi) \stackrel{\text{def}}{=} \frac{D_\Xi u(\xi)-D_\Xi u(\xi^-)}{\xi-\xi^-}, \quad \xi\in\langle y^+, z^-\rangle. \quad (51)$$

Proof. Considering

$$\psi(\xi) \stackrel{\text{def}}{=} \frac{u(\xi^+)-u(\xi)}{\xi^+-\xi}, \quad \xi\in\langle y,z\rangle,$$

we have

$$D_\Xi u(\xi_i) - D_\Xi u(\xi_j) = \psi(\xi_i)-\psi(\xi_j).$$

Applying the formula (38) with $w=\psi$ and knots $\xi_i, \xi_j \in \langle y, z^-\rangle$, we obtain

$$\psi(\xi_i)-\psi(\xi_j) = \mathrm{sgn}(\xi_i-\xi_j)\sum_{p=m(i,j)}^{M(i,j)-1} D_\Xi\psi(\xi_p)(\xi_{p+1}-\xi_p),$$

so that for mentioned knots we find

$$D_\Xi u(\xi_i) - D_\Xi u(\xi_j) = \mathrm{sgn}(\xi_i-\xi_j)\times$$

$$\times\sum_{p=m(i,j)}^{M(i,j)-1}\frac{D_\Xi u(\xi_{p+1})-D_\Xi u(\xi_p)}{\xi_{p+1}-\xi_p}(\xi_{p+1}-\xi_p). \quad (52)$$

Formulas (50) – (51) follow from the relations (39) and (52).

Theorem 4. *If function $\widetilde{u}(t)$ is defined by (34), then*

$$|u(t)-\widetilde{u}(t)| \le (z-y)^2 \max_{\xi\in\langle y^+,z^-\rangle} |D^2_\Xi u(\xi)|, \ t\in\langle y,z\rangle. \quad (53)$$

Proof. Using (50) – (51) under condition $t \in \langle y, z\rangle$, we have

$$|u(t)-\widetilde{u}(t)| \le \sum_{j=0}^{M-1}(\xi_{j+1}-\xi_j)\sum_{i=0}^{k-1}(\xi_{i+1}-\xi_i)\times$$

$$\times\Big|\sum_{p=m(i,j)}^{M(i,j)-1}(\xi_{p+1}-\xi_p)\Big|/(\xi_M-\xi_0)\max_{\xi\in\langle y^+,z^-\rangle}|D^2_\Xi u(\xi)|. \quad (54)$$

The evaluation (53) follows from (54).

Note 3. If it is discussed the sequence of grids (33) in $[y,z]$ with property $\max_{\xi\in\langle y,z^-\rangle}(\xi^+-\xi) \to +0$, then the function $u \in C^1[y,z]$ in the equality (46) gives the evaluation

$$|u(t)-\widetilde{u}(t)| \le (z-y)\max_{\xi\in[y,z]}|u'(\xi)|,$$

and if $u\in C^2[y,z]$ and $t\in[y,z]$, then (53) gives the next enequality

$$|u(t)-\widetilde{u}(t)| \le (z-y)^2\max_{\xi\in[y,z]}|u''(\xi)|.$$

Consider a grid $\widehat{X}$, $\widehat{X}\subset\Xi$ such that ,

$$\widehat{X}: a=\widehat{x}_0<\widehat{x}_1<\widehat{x}_2<\ldots<\widehat{x}_{\widehat{K}}<\widehat{x}_{\widehat{K}+1}=b. \quad (55)$$

Let $u(t)$ be grid function defined on $\langle a,b\rangle$. We construct piecewise linear interpolation

$$\widetilde{u}(t) \stackrel{\text{def}}{=} u(\widehat{x}_j)+\frac{u(\widehat{x}_{j+1})-u(\widehat{x}_j)}{\widehat{x}_{j+1}-\widehat{x}_j}(t-\widehat{x}_j), \quad (56)$$
$$\forall t\in[\widehat{x}_j,\widehat{x}_{j+1}),\ j\in\{0,1,\ldots,\widehat{K}\}.$$

Theorem 5. *If $\widetilde{u}(t)$ is defined by formula (56) on the grid (55) and $t\in\langle\widehat{x}_j,\widehat{x}_{j+1}\rangle$, $\widehat{x}_j^+<\widehat{x}_{j+1}^-$, then the next inequalities are valid*

$$|u(t)-\widetilde{u}(t)| \le (\widehat{x}_{j+1}-\widehat{x}_j)\max_{\xi\in\langle\widehat{x}_j,\,\widehat{x}_{j+1}^-\rangle}|D_\Xi u(\xi)|, \quad (57)$$

$$|u(t)-\widetilde{u}(t)| \le (\widehat{x}_{j+1}-\widehat{x}_j)^2\max_{\xi\in\langle\widehat{x}_j^+,\,\widehat{x}_{j+1}^-\rangle}|D^2_\Xi u(\xi)|. \quad (58)$$

If $u \in C^1[a,b]$*, then*

$$|u(t) - \widetilde{u}(t)| \le \max_{\xi\in[\widehat{x}_j,\,\widehat{x}_{j+1}]} |u'(\xi)|(\widehat{x}_{j+1} - \widehat{x}_j), \quad (59)$$

and if $u \in C^2[a,b]$*, then* $\forall t \in (\widehat{x}_j,\ \widehat{x}_{j+1})$

$$|u(t)-\widetilde{u}(t)| \le \max_{\zeta\in[\widehat{x}_j,\,\widehat{x}_{j+1}]} |u''(\zeta)|(\widehat{x}_{j+1}-\widehat{x}_j)^2. \quad (60)$$

Proof. The evaluations (57) – (58) follow from the inequalities (47) and (53). The relations (59)– (60) follow from passage to limit in (57) – (58) under condition $\max_{\xi\in\langle y,z^-\rangle}(\xi^+ - \xi) \to +0$ by analogy with Note 3.

8 On number of knots for grid of adaptive type

Theorem 6. *If the condition* $|D_\Xi u(t)| \ge c > 0 \quad \forall t \in \langle y,z\rangle$ *is true, and the grid* $\widehat{X}$ *be the same as* $\widetilde{X}(|D_\Xi u(t)|, \eta, \Xi)$*, then*

1) the number $K'_{u,\Xi}(\eta) \stackrel{\text{def}}{=} K(|D_\Xi u(t)|, \eta, \Xi)$ *satisfies to the relations*

$$\sum_{s=0}^{K-1} \max_{t\in\langle\widetilde{x}_s,\,\widetilde{x}_{s+1}\rangle} |D_\Xi u(t)|(\widetilde{x}_{s+1}-\widetilde{x}_s)/\eta \le K'_{u,\Xi}(\eta) <$$

$$< \sum_{s=0}^{K-1} \max_{t\in\langle\widetilde{x}_s,\,\widetilde{x}^+_{s+1}\rangle} |D_\Xi u(t)|(\widetilde{x}^+_{s+1} - \widetilde{x}_s)/\eta, \quad (61)$$

2) the inequality

$$|u(t) - \widetilde{u}(t)| \le \eta \qquad \forall t \in \langle y,z\rangle \quad (62)$$

is true,

3) if $u \in C^1[a,b]$, $|u'(t)| \ge c > 0\ \forall t \in [a,b]$*, and sequence (5.2) satisfies the condition (5.3), then*

$$\lim_{\eta'\to+0}\lim_{\lambda\to+0} K'_{u,\Xi(\lambda)}(\eta')\eta' = \int_a^b |u'(t)|dt. \quad (63)$$

Proof. The formula (61) follows from (7), where $f(t) = |D_\Xi u(t)|$. The inequality (62) can be obtained from (47) and (5), where $f(t) \stackrel{\text{def}}{=} |D_\Xi u(t)|$, $\varepsilon \stackrel{\text{def}}{=} \eta$. The relation (63) follows from (61) by sequential passages to the limits: first we have $\lambda \to +0$, and then we pass η to zero.

Theorem 7. *Suppose the condition*

$$|D^2_\Xi u(t)| \ge c > 0 \qquad \forall t \in \langle y,z\rangle \quad (64)$$

is fulfilled. If the grid $\widehat{X}$ *coincides with* $\widetilde{X}(\sqrt{|D^2_\Xi u(t)|}, \eta, \Xi)$*, then*

1) the quantity of knots $K''_{u,\Xi}(\eta) \stackrel{\text{def}}{=} K(\sqrt{|D^2_\Xi u(t)|}, \eta, \Xi)$ *satisfies to relations*

$$\sum_{s=0}^{K-1} \max_{t\in\langle\widetilde{x}_s,\,\widetilde{x}_{s+1}\rangle} \sqrt{|D^2_\Xi u(t)|}(\widetilde{x}_{s+1}-\widetilde{x}_s)/\eta \le K''_{u,\Xi}(\eta) <$$

$$< \sum_{s=0}^{K-1} \max_{t\in\langle\widetilde{x}_s,\,\widetilde{x}^+_{s+1}\rangle} \sqrt{|D^2_\Xi u(t)|}(\widetilde{x}^+_{s+1} - \widetilde{x}_s)/\eta. \quad (65)$$

2) the inequality

$$|u(t) - \widetilde{u}(t)| \le \eta^2 \qquad \forall t \in \langle y,z\rangle \quad (66)$$

is true,

3) if $u \in C^2[a,b]$, $|u''(t)| \ge c > 0\ \forall t \in [a,b]$*, and (31) is fulfilled, then*

$$\lim_{\eta'\to+0}\lim_{\lambda\to+0} K''_{u,\Xi(\lambda)}(\eta')\eta' = \int_a^b \sqrt{|u''(t)|}dt. \quad (67)$$

Proof. The formula (7.5) follows from (7), where $f(t) = \sqrt{|D^2_\Xi u(t)|}$. The inequality (66) can be obtained from (53) and (5), where $f(t) \stackrel{\text{def}}{=} \sqrt{|D^2_\Xi u(t)|}$, $\varepsilon \stackrel{\text{def}}{=} \eta$. Finally, the formula (67) follows from (65) by sequential passages to the limits (see the proof of Theorem 6).

9 On the quantity of knots of pseudo-equidistant grid

Theorem 8. *If the grid* $\widehat{X}$ *is the same as the grid* $\overline{X}(|D_\Xi u|, \eta, \Xi)$*, then*

1) the number $N'_{u,\Xi}(\eta) \stackrel{\text{def}}{=} N(|D_\Xi u|, \eta, \Xi)$ *of inner knots of the grid satisfies to the relation*

$$(b-a)\|D_\Xi u\|_{C\langle a,b\rangle}/\eta - 1 < N'_{u,\Xi}(\eta) \le$$

$$\le (b-a)\|D_\Xi u\|_{C\langle a,b\rangle}/\eta. \quad (68)$$

2) the inequality

$$|u(t) - \widetilde{u}(t)| \le \eta \qquad \forall t \in \langle a,b\rangle \quad (69)$$

is fulfilled.

Proof. Suppose that $\widehat{X} \stackrel{\text{def}}{=} \overline{X}(|D_\Xi u|, \eta, \Xi)$. Using the formula (26) with $f = |D_\Xi u|$, we get relation (68). The inequality (69) follows from (47) and (27), where $f = |D_\Xi u|$ and $\varepsilon = \eta$.

Theorem 9. *If the grid $\widehat{X}$ is equal to the grid $\overline{X}(\sqrt{|D^2_\Xi u|}, \eta, \Xi)$, then*

1) the number $N''_{u,\Xi}(\eta) \stackrel{\text{def}}{=} N(\sqrt{|D^2_\Xi u|}, \eta, \Xi)$ of inner knots of the grid satisfies to relation

$$(b-a)\| \, |D^2_\Xi u|^{1/2}\|_{C\langle a,b\rangle}/\eta - 1 < N''_{u,\Xi}(\eta) \leq$$
$$\leq (b-a)\| \, |D^2_\Xi u|^{1/2}\|_{C\langle a,b\rangle}/\eta, \qquad (70)$$

2) the inequality

$$|u(t) - \widetilde{u}(t)| \leq \eta^2 \qquad \forall t \in \langle a, b\rangle \qquad (71)$$

is true.

Proof. Applying the formula (26) with $\widehat{X} \stackrel{\text{def}}{=} \overline{X}(\sqrt{|D^2_\Xi u|}, \eta, \Xi)$, $f = |D^2_\Xi u|$, we get the relation (70). The inequality (71) follows from (53) and (27) if $f = \sqrt{|D^2_\Xi u|}$ $\varepsilon = \eta$.

10 Relative characteristic of the quantities of knots for different grids under condition of the same approximation

Theorem 10. *The inequality $|\widetilde{u}(t) - u(t)| \leq \eta$ is true for each of two variants of grids: $\widehat{X} \stackrel{\text{def}}{=} \overline{X}(|D_\Xi u|, \eta, \Xi)$ and $\widehat{X} \stackrel{\text{def}}{=} \widetilde{X}(|D_\Xi u|, \eta, \Xi)$. In addition we have*

$$\frac{(b-a)\|D_\Xi u\|_{C\langle a,b\rangle} - \eta}{\sum\limits_{s=0}^{K-1} \max\limits_{t\in\langle \widetilde{x}_s,\, \widetilde{x}^+_{s+1}\rangle} |D_\Xi u(t)|(\widetilde{x}^+_{s+1} - \widetilde{x}_s)} < \frac{N'_{u,\Xi}(\eta)}{K'_{u,\Xi}(\eta)} \leq$$
$$\leq \frac{(b-a)\|D_\Xi u\|_{C\langle a,b\rangle}}{\sum\limits_{s=0}^{K-1} \max\limits_{t\in\langle \widetilde{x}_s,\, \widetilde{x}_{s+1}\rangle} |D_\Xi u(t)|(\widetilde{x}_{s+1} - \widetilde{x}_s)}. \qquad (72)$$

Proof. Using the inequality (28), we put $f = |D_\Xi u|$ and $\varepsilon = \eta$. As a result we get (72).

Theorem 11. *If the family of grids (30) has property (31), $u \in C^1[a,b]$ and $\|u'\|_{C[a,b]} \neq 0$, then*

$$\lim_{\eta\to+0}\lim_{\lambda\to+0} \frac{K'_{u,\Xi(\lambda)}(\eta)}{N'_{u,\Xi(\lambda)}(\eta)} = \frac{\frac{1}{b-a}\int_a^b |u'(t)|dt}{\|u'\|_{C[a,b]}}. \qquad (73)$$

Proof. Passing on to the limit $\lambda \to +0$ in the relation (72) we have

$$\frac{(b-a)\| \, u'\|_{C[a,b]} - \eta}{\int_a^b |u'(t)|dt} < \lim_{\lambda\to+0} \frac{N'_{u,\Xi(\lambda)}(\eta)}{K'_{u,\Xi(\lambda)}(\eta)} \leq$$
$$\leq \frac{(b-a)\| \, u'\|_{C[a,b]}}{\int_a^b |u'(t)|dt},$$

and passing on to the limit $\eta \to +0$, we get (73).

Theorem 12. *If $\widehat{X} \stackrel{\text{def}}{=} \overline{X}(\sqrt{|D^2_\Xi u|}, \eta, \Xi)$ or if $\widehat{X} \stackrel{\text{def}}{=} \widetilde{X}(\sqrt{|D^2_\Xi u|}, \eta, \Xi)$, then in the both cases the evaluation $|\widetilde{u}(t) - u(t)| \leq \eta^2$ is correct. Moreover we have*

$$\frac{(b-a)\| \, |D^2_\Xi u|^{1/2}\|_{C\langle a,b\rangle} - \eta}{\sum\limits_{s=0}^{K-1} \max\limits_{t\in\langle \widetilde{x}_s,\, \widetilde{x}^+_{s+1}\rangle} \sqrt{|D^2_\Xi u(t)|}(\widetilde{x}^+_{s+1} - \widetilde{x}_s)} < \frac{N''_{u,\Xi}(\eta)}{K''_{u,\Xi}(\eta)} \leq$$
$$\leq \frac{(b-a)\| \, |D^2_\Xi u|^{1/2}\|_{C\langle a,b\rangle}}{\sum\limits_{s=0}^{K-1} \max\limits_{t\in\langle \widetilde{x}_s,\, \widetilde{x}_{s+1}\rangle} \sqrt{|D^2_\Xi u(t)|}(\widetilde{x}_{s+1} - \widetilde{x}_s)}. \qquad (74)$$

Proof. Applying the inequality (28) with $f = \sqrt{|D^2_\Xi u|}$ and $\varepsilon = \eta$, we get (74).

Theorem 13. *If the family of grids (30) has the property (31) and the function $u \in C^2[a,b]$ satisfies to the relation $\|u''\|_{C[a,b]} \neq 0$, then*

$$\lim_{\eta\to+0}\lim_{\lambda\to+0} \frac{K''_{u,\Xi(\lambda)}(\eta)}{N''_{u,\Xi(\lambda)}(\eta)} = \frac{\frac{1}{b-a}\int_a^b \sqrt{|u''(t)|}dt}{\|\sqrt{|u''|}\|_{C[a,b]}}. \qquad (75)$$

Proof. Passing on to the limit $\lambda \to +0$ in the relation (74) we have

$$\frac{(b-a)\| \, |u''|^{1/2}\|_{C[a,b]} - \eta}{\int_a^b \sqrt{|u''(t)|}dt} < \lim_{\lambda\to+0} \frac{N''_{u,\Xi(\lambda)}(\eta)}{K''_{u,\Xi(\lambda)}(\eta)} \leq$$
$$\leq \frac{(b-a)\| \, |u''|^{1/2}\|_{C[a,b]}}{\int_a^b \sqrt{|u''(t)|}dt};$$

now passing on to the limit $\eta \to +0$, we obtain (75).

11 Conclusion

In the offered paper the algorithm of approximation for discrete function (discrete number flow) is proposed for the splines of the first order associated with the suggested adaptive grid.

We would be very interested in at least the two approaches to processing of the numerical flow: 1) usage of splines of zero order (i.e. piecewise-constant functions), 2) usage of splines of the second order. In the first approach we would obtain the simple approximation algorithm with small quantity arithmetical operations, but the second approach would give the high order of approximation for "smooth" discrete number flows.

The both approaches will be discussed later in connection with spline-wavelet decompositions.

In directions of our future researches we plan to obtain the adaptive schemes of algorithms for wavelet decompositions of discrete number flows, to discuss their properties and to demonstrate their opportunities in practical aspects.

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