Power-Efficient Linear Phase FIR Notch Filter Design Using the LARS Scheme

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Abstract: In this paper, an effective paradigm based on the Least Angle Regression (LARS) scheme is developed to iteratively compute the power-efficient linear phase FIR notch filters. At each iteration, we compute the equiangular vector and the step size to be taken which are then used to modify our previous prediction of the filter coefficients along the computed equiangular direction. The iteration of the LARS scheme stops when the error defined as the L_2-norm of the difference between the computed prediction of the notch filter and the desired notch filter is less than the pre-chosen design error ϵ. The simulation results demonstrate that the proposed LARS scheme is an effective paradigm to compute power-efficient linear phase notch FIR filters.

Key–Words: Least angle regression scheme, notch filter, FIR filter, linear phase, power-efficient,filter design

1 Introduction

The power-efficient design of the digital FIR filters is always of interest due to its benefits not only in terms of computation but also other cost measurements such as hardware and energy consumption. The power-efficiency determined by the arithmetic operations of the digital filters can be increased by designing filters with fewer non-zero coefficients. Therefore, it is essential to study the power-efficient design of the linear phase FIR notch filters which have widespread applications in communication systems, radar systems and biomedical signal processing [1, 2].

In the past few decades, many different methods of designing linear phase FIR notch filters such as the maximally flat method, semi-analytic method, and multiple exchange algorithm have been reported [3]–[10]. Of these approaches, the precise equiripple (PE) [6] and equiripple method (ER) [8] based on the expansion of the generating polynomials into the Chebyshev polynomials to compute the impulse responses offer us the efficient implementation of linear phase notch filters. More recently, a novel paradigm [10] utilizes the orthogonal matching pursuit (OMP) scheme to iteratively compute the power-efficient approximation of the frequency response of a desired notch filter.

The Least Angle Regression (LARS) scheme [11] is a method which stems from statistics, and has been shown to be successful in solving variable selection problems. In this brief, an effective paradigm based on the LARS scheme is developed to iteratively compute the power-efficient linear phase FIR notch filters. At each iteration, we compute the equiangular vector and the step size to be taken which are then used to modify our previous prediction of the filter coefficients along the computed equiangular direction. By moving along the equiangular direction, the correlation between each correlated covariate (a vector comprised of complex exponentials) and the residual vector of the desired notch filter and computed fitted filter decreases at an equal speed. The iteration stops when the error defined as the L_2-norm of the difference between the desired notch filter and the computed approximation of it is less than the pre-chosen $\epsilon > 0$. The simulation results demonstrate that under the similar performance, the linear phase notch FIR filters yielded from the scheme based LARS algorithm are more power-efficient than those generated from the competing methods.

The rest of this paper is organized as follows: The FIR notch filter design problem is formulated in Section 2. In Section 3, the procedure of computing the FIR notch filter based on the LARS scheme is illustrated. The simulation results are presented in Section 4.

2 The Problem

From [8], the typical specifications associated with the notch filters include the notch frequency ω_s, rejection bandwidth $\overline{BW}$ and pass band ripple δ, as shown in

Fig. 1.

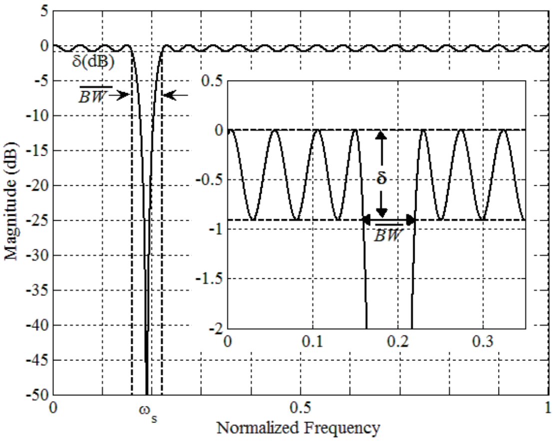

Figure 1: The specification of a digital notch filter

We assume that an N-th order linear phase FIR notch filter $H(e^{j\omega})$ is represented as

$$H(e^{j\omega}) = \sum_{n=0}^{N} h_n e^{-j\omega n} \quad (1)$$

where h_n ($0 \le n \le N$) is the impulse response of $H(e^{j\omega})$. To simplify the derivation, we assume that the filter (1) is Type I FIR filter, i.e., the order N of (1) is even and its tap weighs satisfy $h_n = h_{N-n}$ for all $0 \le n \le N$ throughout the rest of this paper.

Utilizing Type I (linear phase) assumption, the filter (1) can be represented as

$$H(e^{j\omega}) = e^{-jM\omega} H_{\text{zero}}(e^{j\omega}) \quad (2)$$

where $H_{\text{zero}}(e^{j\omega})$, the zero-phase of (1), is defined as

$$\begin{aligned} H_{\text{zero}}(e^{j\omega}) &= h_M + 2\sum_{m=1}^{M} h_{M-m}\cos(m\omega) \\ &= (1,\cos(\omega),\cdots,\cos(M\omega))(h_M, 2h_{M-1},\cdots,2h_0)^{\mathrm{T}} \end{aligned} \quad (3)$$

with $M = N/2$.

Given $\varepsilon > 0$, our goal is to develop a procedure of computing power-efficient linear phase FIR notch filters (2) whose zero-phase $H_{\text{zero}}(e^{j\omega})$ satisfies

$$\|H_{\text{zero}}(e^{j\omega}) - H_{\text{d}}(e^{j\omega})\|_2 < \varepsilon \ , \ \ \forall \omega \in [0, \pi] \quad (4)$$

where $H_{\text{d}}(e^{j\omega})$ is the desired filter frequency response. To compute a solution of problem (4), we follow the standard discretization procedure as presented in [12] and replace the continuous parameter ω by $L+1$ samples (where $L \gg 1$ is a large positive integer) uniformly distributed in the frequency set $[0, \pi]$. Thus, the discretization and normalized formulation of problem (4) is expressed as

$$\|\mathbf{A}\mathbf{x} - \mathbf{f}\|_2 < \varepsilon \,, \quad (5)$$

where we have

$$\mathbf{A} = (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_m, \ldots, \mathbf{a}_M) \quad (6)$$

$$\mathbf{a}_m = (1, \cos(m\omega_0), \cos(2m\omega_0), \ldots, \cos(Lm\omega_0))^{\mathrm{T}} \quad (7)$$

$$\mathbf{x} = (\, h_M \,,\, 2h_{M-1} \,, \cdots ,\, 2h_0 \,)^{\mathrm{T}} \quad (8)$$

$$\mathbf{f} = (\, H_{\text{d}}(e^{j0}), H_{\text{d}}(e^{j\omega_0}), \cdots , H_{\text{d}}(e^{jL\omega_0})\,)^{\mathrm{T}} \quad (9)$$

with $\omega_0 = \pi/L$, $0 \le l \le L$ and $0 \le m \le M$ ($M = N/2$). The symbol $\|\bullet\|_2$ represents the L_2-norm of the vector.

3 Linear Phase Notch Filters

In this section, we utilize the LARS scheme to develop a procedure of computing the solution of problem (5). Given the design specifications, i.e., δ (passband ripple), $\overline{BW}$ (rejection bandwidth) and ω_s (notch frequency), the frequency response of the desired notch filter $H_{\text{d}}(e^{j\omega})$ is

$$H_{\text{d}}(e^{j\omega}) = \begin{cases} 0; & \text{if } \ \omega_s - \Delta F \le \omega \le \omega_s + \Delta F \\ 1; & \text{otherwise} \end{cases} \quad (10)$$

with $\Delta F < \overline{BW}/2$.

To begin with, let us first estimate the order N of the initial filter (3) through

$$N = \max\{\hat{N}_4(\omega_{p_1}, \Delta F, \delta, \delta_s), \hat{N}_4(\omega_{p_2}, \Delta F, \delta, \delta_s)\} \quad (11)$$

where $\hat{N}_4(\cdot)$ is determined by equation [13, eq.(20)]. The arguments of $\hat{N}_4(\cdot)$ in (11) are equal to

$$\omega_{p_1} = \omega_s - \Delta F \quad (12)$$

$$\omega_{p_2} = 1 - \omega_s + \Delta F \quad (13)$$

$$\delta_s = 10^{\frac{A_{\text{notch}}}{20}} \,, \quad (14)$$

where the negative number A_{notch} expressed in decibels represents the attenuation at the notch frequency.

The procedure of computing the linear phase notch filters proceeds through the following steps:

Step 1**:** Choose the initial active set $\mathcal{I}_0 = \phi$, the initial prediction vector $\hat{\mathbf{f}}_0 = 0$ and the initial $(M+1) \times 1$ vector

$$\hat{\mathbf{x}}^{(0)} = (\hat{x}_0^{(0)} \,,\, \hat{x}_1^{(0)} \,, \ldots ,\, \hat{x}_M^{(0)})^{\mathrm{T}} = (0\,, \ldots, 0)^{\mathrm{T}} \quad (15)$$

Step 2**:** Compute the vector of the correlations between the columns of $\mathbf{A}$ defined in (6) and the initial residual vector $\mathbf{r}_0 = \mathbf{f} - \hat{\mathbf{f}}_0$ as

$$\hat{\mathbf{C}}^{(0)} = \mathbf{A}^{\mathrm{T}}(\mathbf{f} - \hat{\mathbf{f}}_0) = (\hat{c}_0^{(0)}, \hat{c}_1^{(0)}, \cdots, \hat{c}_M^{(0)})^{\mathrm{T}} \quad (16)$$

with $\mathbf{f}$ being given in (9). Find the index $j_1 \in \{0, 1, \cdots, M\}$ satisfying

$$j_1 = \arg\max_{j \in \mathcal{I}_0^c} \{|\hat{c}_j^{(0)}|\} \quad (17)$$

where we have $\mathcal{I}_0^c = \{0, 1, \ldots, M\} - \mathcal{I}_0$, and the new active set is updated as $\mathcal{I}_1 = \mathcal{I}_0 \bigcup \{j_1\} = \{j_1\}$.

Step 3**:** Using (6), (16) and (17), compute

$$\hat{\gamma}_1 = \min_{j \in \mathcal{I}_1^c}{}^{+}\{\frac{\hat{c}_{j_1}^{(0)} - \hat{c}_j^{(0)}}{1 - \mathbf{a}_j^{\mathrm{T}} \mathbf{a}_{j_1}}, \frac{\hat{c}_{j_1}^{(0)} + \hat{c}_j^{(0)}}{1 + \mathbf{a}_j^{\mathrm{T}} \mathbf{a}_{j_1}}\} \quad (18)$$

$$j_2 = \arg\min_{j \in \mathcal{I}_1^c}{}^{+}\{\frac{\hat{c}_{j_1}^{(0)} - \hat{c}_j^{(0)}}{1 - \mathbf{a}_j^{\mathrm{T}} \mathbf{a}_{j_1}}, \frac{\hat{c}_{j_1}^{(0)} + \hat{c}_j^{(0)}}{1 + \mathbf{a}_j^{\mathrm{T}} \mathbf{a}_{j_1}}\} \quad (19)$$

with $\mathcal{I}_1^c = \{0, 1, \ldots, M\} - \mathcal{I}_1$. In (18) and (19), the symbol $\min^{+}$ denotes that the minimum is taken over only positive components with each choice of $j \in \mathcal{I}_1^c$. From the LARS scheme [11], the prediction vector $\hat{\mathbf{f}}_1$ at the first iteration is therefore equal to

$$\hat{\mathbf{f}}_1 = \hat{\mathbf{f}}_0 + \hat{\gamma}_1 \mathbf{a}_{j_1}. \quad (20)$$

Update the new active set as $\mathcal{I}_2 = \mathcal{I}_1 \bigcup \{j_2\} = \{j_1, j_2\}$. From [11], it follows $|\mathbf{a}_{j_1}^{\mathrm{T}} \mathbf{r}_1| = |\mathbf{a}_{j_2}^{\mathrm{T}} \mathbf{r}_1|$ where the residual vector $\mathbf{r}_1 = \mathbf{f} - \hat{\mathbf{f}}_1$.

Using (15) and (20), we can compute the new vector $\hat{\mathbf{x}}^{(1)} = (\hat{x}_0^{(1)}, \hat{x}_1^{(1)}, \ldots, \hat{x}_M^{(1)})^{\mathrm{T}}$ whose entries are equal to

$$\hat{x}_j^{(1)} = \begin{cases} \hat{x}_j^{(0)} + \hat{\gamma}_1; & \text{if } j = j_1 \\ \hat{x}_j^{(0)}; & \text{otherwise} \end{cases}. \quad (21)$$

From (8), the tap weights of the notch filter $\{h_m\}_{m=0}^{M}$ can be computed from the vector $\hat{\mathbf{x}}^{(1)}$.

Step 4**:** For $2 \le k \le M + 1$, our design procedure proceeds as follows:

1. Compute the vector of the correlation between the columns of $\mathbf{A}$ and $\mathbf{r}_{k-1} = \mathbf{f} - \hat{\mathbf{f}}_{k-1}$ as

$$\hat{\mathbf{C}}^{(k-1)} = \mathbf{A}^{\mathrm{T}} \mathbf{r}_{k-1} = (\hat{c}_0^{(k-1)}, \hat{c}_1^{(k-1)}, \cdots, \hat{c}_M^{(k-1)})^{\mathrm{T}} \quad (22)$$

2. To employ the LARS scheme [11] for solving (5), we introduce the equiangular vector $\mathbf{u}_k$ defined as

$$\mathbf{u}_k = \mathbb{A}_k \mathbf{w}_k \quad (23)$$

where we have

$$\mathbf{w}_k = (w_1, w_2, \ldots, w_k)^{\mathrm{T}} = \alpha_k (\mathbb{A}_k^{\mathrm{T}} \mathbb{A}_k)^{-1} \mathbf{1}_k \quad (24)$$

$$\alpha_k = (\mathbf{1}_k^{\mathrm{T}} \mathbb{A}_k^{\mathrm{T}} \mathbb{A}_k \mathbf{1}_k)^{-\frac{1}{2}}. \quad (25)$$

$\mathbf{1}_k = (1, \ldots, 1)^{\mathrm{T}}$ is a $k \times 1$ vector. The matrix $\mathbb{A}_k$ is comprised of all the column vectors $\mathbf{a}_j$ from $\mathbf{A}$ defined in (6) with $j \in \mathcal{I}_k$, i.e.,

$$\mathbb{A}_k = (s_{j_1} \mathbf{a}_{j_1}, \ldots, s_{j_k} \mathbf{a}_{j_k}) \quad (26)$$

where from (22) s_j is defined as

$$s_j = \text{sign}\{\hat{c}_j^{(k-1)}\} = \begin{cases} 1; & \text{if } \hat{c}_j^{(k-1)} \ge 0 \\ -1; & \text{if } \hat{c}_j^{(k-1)} < 0 \end{cases} \quad (27)$$

3. Using (22), compute

$$\hat{c} = \max_{0 \le j \le M} \{|\hat{c}_j^{(k-1)}|\}. \quad (28)$$

Along the equiangular vector $\mathbf{u}_k$ defined in (23), the LARS scheme [11] computes $\hat{\mathbf{f}}_k$, $\hat{\gamma}_k$ and j_{k+1} through

$$\hat{\mathbf{f}}_k = \hat{\mathbf{f}}_{k-1} + \hat{\gamma}_k \mathbf{u}_k \quad (29)$$

$$\hat{\gamma}_k = \min_{j \in \mathcal{I}_k^c}{}^{+}\{\frac{\hat{c} - \hat{c}_j^{(k-1)}}{\alpha_k - b_j}, \frac{\hat{c} + \hat{c}_j^{(k-1)}}{\alpha_k + b_j}\} \quad (30)$$

$$j_{k+1} = \arg\min_{j \in \mathcal{I}_k^c}{}^{+}\{\frac{\hat{c} - \hat{c}_j^{(k-1)}}{\alpha_k - b_j}, \frac{\hat{c} + \hat{c}_j^{(k-1)}}{\alpha_k + b_j}\} \quad (31)$$

with $b_j = \mathbf{a}_j^{\mathrm{T}} \mathbf{u}_k$ and α_k defined in (25).

4. Update the new active set as $\mathcal{I}_{k+1} = \mathcal{I}_k \bigcup \{j_{k+1}\}$.

5. From (29), the vector $\hat{\mathbf{x}}^{(k)} = (\hat{x}_0^{(k)}, \ldots, \hat{x}_M^{(k)})^{\mathrm{T}}$ can be computed as follows:

• For $j \in \mathcal{I}_k = \{j_1, j_2, \cdots, j_n, \cdots, j_k\}$, we have

$$\hat{x}_{j_n}^{(k)} = \hat{x}_{j_n}^{(k-1)} + \hat{\gamma}_k * s_{j_n} * w_n \quad (32)$$

where w_n is the n-th entry in the vector $\mathbf{w}_k$ defined in (24) for $1 \le n \le k$.

• For $j \notin \mathcal{I}_k$, we have

$$\hat{x}_j^{(k)} = \hat{x}_j^{(k-1)}. \quad (33)$$

From (8), the tap weights of the notch filter $\{h_m\}_{m=0}^{M}$ can be computed from the vector $\hat{\mathbf{x}}^{(k)}$.

Step 5**:** Using (9) and (29), compute the error ε_k at the k-th iteration as

$$\varepsilon_k = \|\hat{\mathbf{f}}_k - \mathbf{f}\|_2. \quad (34)$$

If ε_k is less than the pre-given design error ε, then the vector $\hat{\mathbf{x}}^{(k)}$ yields the tap weights of the power-efficient linear phase notch filter that meets our design objective of (4). Otherwise, the same procedure as described in Step 4 is repeated.

Table 1: Performance of the LARS Scheme Compared to the OMP and ER Schemes in Example 1

Design method	Filter order	Number of nonzero taps	Rejection bandwidth	Passband ripple	Attenuation at notch frequency
LARS	76	63	0.0610π	-0.9322dB	-305.2dB
OMP	86	65	0.0610π	-0.9394dB	-312.1dB
ER	76	77	0.0607π	-0.9109dB	-308.9dB

Table 2: Impulse Response of the Notch Filters

Example 1			Example 2		
n		h_n	n		h_n
0	76	-0.0204001	0	72	0.0139436
1	75	0.0174606	1	71	0.0064912
2	74	-0.0102290	2	70	-0.0044051
3	73	0	3	69	-0.0110903
4	72	0	4	68	-0.0106608
5	71	-0.0114275	5	67	0
6	70	0.0132315	6	66	0.0097042
7	69	-0.0179306	7	65	0.0150598
8	68	0.0122499	8	64	0
9	67	-0.0079699	9	63	-0.0105411
10	66	0	10	62	-0.0193027
11	65	0.0085097	11	61	-0.0117056
12	64	-0.0188880	12	60	0.0102747
13	63	0.0208866	13	59	0.0218423
14	62	-0.0211890	14	58	0.0158759
15	61	0.0117662	15	57	-0.0049704
16	60	0	16	56	-0.0248972
17	59	-0.0082352	17	55	-0.0232417
18	58	0.0234272	18	54	0
19	57	-0.0230846	19	53	0.0241777
20	56	0.0283422	20	52	0.0319756
21	55	-0.0224942	21	51	0.0098404
22	54	0	22	50	-0.0220426
23	53	0.0074522	23	49	-0.0338544
24	52	-0.0241713	24	48	-0.0195978
25	51	0.0295327	25	47	0.0155936
26	50	-0.0326831	26	46	0.0384234
27	49	0.0248730	27	45	0.0297088
28	48	-0.0091551	28	44	-0.0053761
29	47	0	29	43	-0.0355259
30	46	0.0265334	30	42	-0.0361281
31	45	-0.0314075	31	41	0
32	44	0.0346608	32	40	0.0303661
33	43	-0.0292619	33	39	0.0437485
34	42	0.0118822	34	38	0.0154423
35	41	0	35	37	-0.0231322
36	40	-0.0209545	36		0.9933402
37	39	0.0313320			
38		0.9935895			

4 Numerical Examples

In this section, we employ the LARS scheme developed in the previous section to design the power-efficient linear phase FIR notch filters. As a comparison, under the same design specifications, we also compute the notch filters through the OMP method [10], the optimal equiripple method (ER) [8] and the precise equiripple method (PE) [6] . The simulation results demonstrate that the proposed LARS scheme is an effective paradigm to compute power-efficient linear phase notch FIR filters.

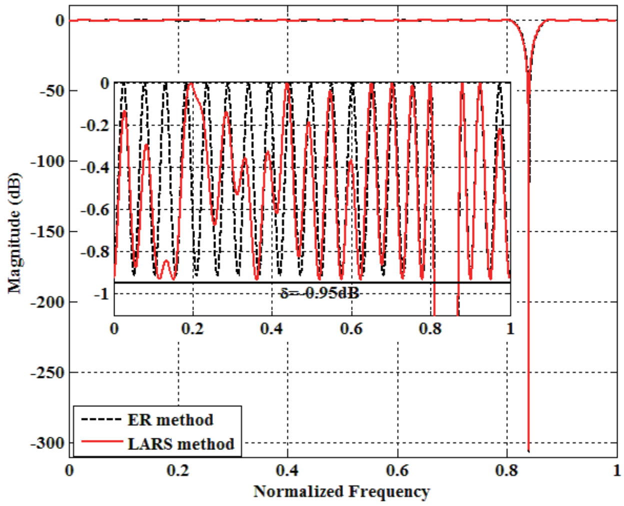

Figure 2: Magnitude frequency responses of the notch filters derived from the LARS scheme(solid line) and ER method(dashed line).

Example 1: Consider the following linear phase notch filter design problem: Compute the notch filter subject to 0.061π (the rejection bandwidth) for -0.95dB (the passband ripple) and 0.84π (the notch frequency), which is identical to the specifications of [8].

Choose the attenuation at the notch frequency $A_{\text{notch}} = -120$dB. Substituting this choice and the design specifications into (11), we obtain $N = 92$ (the order of the initial filter). Select $\varepsilon = 0.04$ defined in (4).

In TABLE 1 , the filter orders, the numbers of the nonzero tap weights, rejection bandwidths, passband

Table 3: Performance of the LARS Scheme Compared to the OMP and PE Schemes in Example 2

Design method	Filter order	Number of nonzero taps	Rejection bandwidth	Passband ripple	Attenuation at notch frequency
LARS	72	65	0.0759π	-0.4976dB	-316.0dB
OMP	80	65	0.0736π	-0.4665dB	-309.5dB
PE	72	73	0.0780π	-0.4613dB	-310.0dB

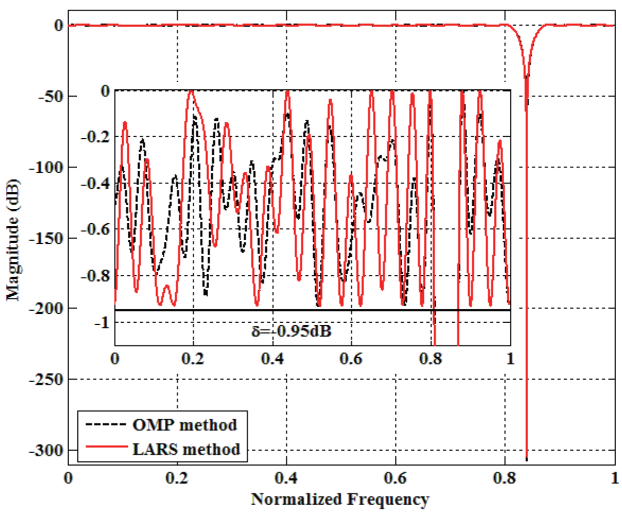

Figure 3: Magnitude frequency responses of the notch filters derived from the LARS scheme(solid line) and OMP method(dashed line).

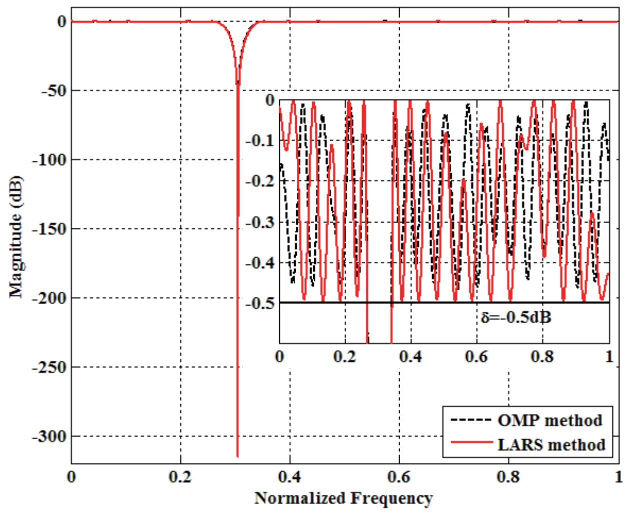

Figure 5: Magnitude frequency responses of the notch filters derived from the LARS scheme(solid line) and OMP method(dashed line).

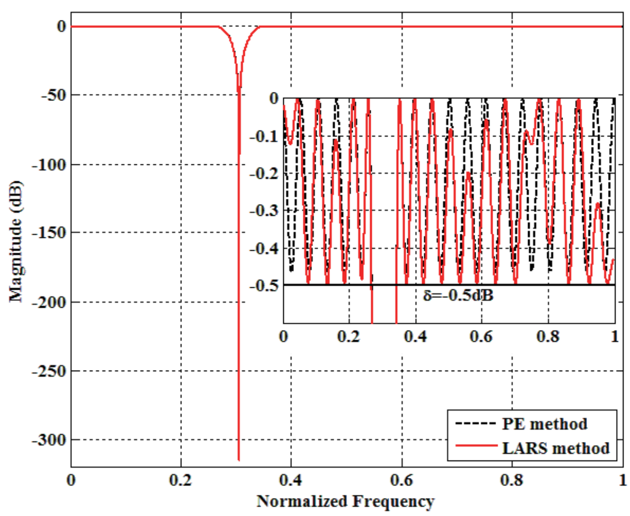

Figure 4: Magnitude frequency responses of the notch filters derived from the LARS scheme(solid line) and PE method(dashed line).

ripples and attenuations at the notch frequency corresponding to the LARS, OMP [10] and ER schemes [8] are presented. The tap weights of the linear phase FIR notch filter derived from the LARS scheme are listed on the left part of TABLE 2. The magnitudes of frequency responses of the LARS scheme compared to the OMP and ER schemes are shown in Fig. 2 and Fig. 3 respectively.

Example 2: Consider the design specifications given in [6], i.e., 0.075π (the rejection bandwidth) for -0.5dB (the passband ripple), 0.3π (the notch frequency).

Choose the attenuation at the notch frequency $A_{\text{notch}} = -120$dB. Substituting this choice and the design specifications into (11), we obtain $N = 86$ (the order of the initial filter). Set the design error $\varepsilon = 0.02$ of (4).

The tap weights of the notch filter computed through the LARS scheme are listed in the right part of TABLE 2. In TABLE 3 , the filter orders, numbers of the nonzero tap weights, rejection bandwidths (for $\delta = -0.5$dB), passband ripples and attenuations at the notch frequency corresponding to the LARS, OMP and PE [6] schemes are presented respectively. The magnitudes of frequency responses of the LARS scheme compared to the OMP and ER schemes are shown in Fig. 4 and Fig. 5 respectively.

5 Conclusion

In this paper, an effective paradigm based on the Least Angle Regression (LARS) scheme is developed to iteratively compute the power-efficient linear phase FIR notch filters. At each iteration, we compute the equiangular vector and the step size to be taken which are then used to modify our previous prediction of the filter coefficients along the computed equiangular direction. The iteration of the LARS scheme stops when the refined filter meets the predevised design specifications. The simulation results demonstrate that the proposed LARS scheme is an effective paradigm to compute power-efficient linear phase notch FIR filters.

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