Kalman Filter Design for Time Delay Systems

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Abstract: - In this paper, an observer design is proposed for linear time delay systems. An easy way to compute least square estimation error of an observer for time delay systems is derived, where the time delay terms exist in the state and output of the system. Based on the least square estimation error an optimization algorithm to compute a Kalman filter for time delay systems is proposed. By employing the finite characterization of a Lyapunov functional equation, the existence of sufficient conditions for obtaining the right solution and guaranteeing the proper convergence rate of the estimation error has been evaluated. It will be shown that this finite characterization can be calculated by means of a matrix exponential function. The desirable performance of the proposed observer has been demonstrated through the simulation of several numerical examples.

Key-Words: - time delay system,state delay system, observer, Kalman filter

1 Introduction

As a dual of the control problem, the state estimation or filtering problem is of great importance in both theory and application, and in the last decades , this problem has gotten extensive concern and many solution schemes have been proposed and successfully put into action. Among them, Kalman filtering, which minimizes the variance of the estimation error, is the most famous one [11].

Kalman filtering is one of the most popular estimation approaches. This filtering method assures that both the state equation and output measurement are subjected to stationary Gaussian noises. The applications of the Kalman filtering theory may be found in a large spectrum of different fields ranging from various engineering problems to biology, geoscience, economics, and management, etc.

A dynamic system whose state variables are estimations of the state variables of another system is called the observer of that system. This expression was first introduced in 1963 into the theory of linear systems by Luenberger [1]. He showed that for every observable linear system, an observer can be designed whose estimation error (i.e. the difference between the real state of the system and the observer state) becomes zero at every considered speed. In fact, an observer is a dynamic system whose inputs are the process inputs and outputs, and whose outputs are the estimated state variables. It can be stated that an estimator of state is an indispensible member of the control systems theory, and it has important applications in feedback control, system supervision and in the fault diagnosis of dynamic systems.

In the control process, it is often assumed that the internal state vectors exist and are available in the measurement of the output; while in practice, this is not the case, and it is necessary to devise an observer in order to provide an estimation of state vectors. If the estimation and reconstruction of all the state variables is needed, the full-order observers, and if the estimation and reconstruction of a number of state variables is needed, the reduced-order observers are used. Time-delayed systems play significant roles in theoretical as well as practical fields; and this influence can be observed in numerous research articles written on various problems that involve this class of systems [2-8]. During the last decade, the theory of observer design for time delay systems has been widely contemplated [28-34]. The estimation of state variables is an important dynamic model, which adds to our knowledge of different systems and helps us analyze and design various controllers. Different approaches have been used for the designing of observers, including: the coordinate change approach [9], the LMI method [10], reducing transformation technique [11], factorization approach [12], polynomial approach [13], modal observer [14], reduced-order observer [15] and the output injection based observer [16]. In

[17], through an algebraic approach, an observer with delay-independent stability for systems with one output delay has been presented. In [18], an observer has been proposed that uses the H_∞ norm as the performance index. The H_∞ filter has been considered in [19], [20] by applying the delay independent stability conditions, in which the matrix inequality has been used. We also frequently encounter the issue of state delay in control problems and physical systems. In recent years, the systems with delay in state have attracted the attention of many researchers, and numerous approaches have been proposed for the evaluation of stability in these systems (see [21] and [22] and the references cited in them).

The goal of this article is to design an observer for time delay systems in which the time delay terms exist in the output and in the state variables, and also the inputs are mixed with noise and the system output accompanies measurement noises. An easy way to compute least square estimation error of an observer for time delay system is derived. This least square estimation error coincides with that of a Kalman filter when time delay is zero. Based on the least square estimation error, an optimization algorithm to compute an observer is proposed.

In the designing of this observer we have used the H_2 norm as the performance index. However, despite the usefulness of the H_2 norm, few observers have used it as the performance index. In [23] and [24] a method has been proposed for the calculation of the H_2 norm of time delay systems by means of the delay Lyapunov equation. In [25], an observer has been offered for time delay systems by applying the delay-independent stability conditions. It should be mentioned that delay-independent approaches are generally more conservative than delay-dependent ones. In this article, for the estimation of system states, a Kalman filter has been proposed whose design uses the delay-dependent stability conditions. Note that when there are no time delay terms, observer is a standard Kalman filter. The optimal observer will be designed by employing the finite characterization of a Lyapunov functional equation as a matrix exponential function and applying the unconstrained nonlinear optimization algorithm. Finally, the proposed observer in this article will be used to estimate the current states based on the time delay system, where the time delay terms exist in the state and in the output of the system.

This article has been organized in the following manner. In section 2, for the definition of the observer, the necessary mathematics has been presented. In section 3, the calculation of the H_2 norm has been offered for the state delay system. In section 4, the method of filter design has been described. In section 5, in order to test the practical usefulness of the proposed technique, it has been applied for solving the estimation problem of several linear systems with time delay. And finally, the summary and conclusion of the obtained results have been presented in the last section.

2 Problem Formulation And Assumptions

Consider linear time-invariant systems described by

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-h) + B_1\omega(t) + B_2 u(t) \\ y(t) &= C_0 x(t) + C_1 x(t-h) + C_2 \nu(t) \end{aligned} \tag{1}$$

Where $x \in R^n$ is the state, $\omega \in R^p$ is the process noise, $u \in R^q$ is the input, $y \in R^r$ is the measurement, and $\nu \in R^r$ is the measurement noise. The h is constant known time delay in the states and the outputs.

It is assumed that ν and ω are uncorrelated white Gaussian processes, which satisfy

$$\begin{aligned} E\{\omega(t)\} = 0, E\{\omega(t)\omega(s)'\} = I\,\delta(t-s) \\ E\{\nu(t)\} = 0, E\{\nu(t)\nu(s)'\} = I\,\delta(t-s) \end{aligned} \tag{2}$$

The objective of this paper is to derive a Kalman filter for time delay system (l), where a filter has the following form:

$$\dot{\hat{x}}(t) = G\hat{x}(t) + Ky(t) + B_2 u(t) \tag{3}$$

Defining the estimation error $e(t)$ as

$$e(t) \triangleq x(t) - \hat{x}(t)$$

From (1) and (3) , we have

$$\begin{aligned} \dot{e}(t) = (A_0 - G - KC_0)x(t) + Ge(t) + \\ \left(A_1 - KC_1\right)x(t-h) + B_1\omega(t) - KC_2\nu(t) \end{aligned} \tag{4}$$

And the augmented system with (1) is given by

$$G_a : \begin{aligned} \dot{\eta}(t) &= \bar{A}_0\eta(t) + \bar{A}_1\eta(t-h) + B\,\zeta(t) \\ e(t) &= C\,\eta(t) \end{aligned} \tag{5}$$

where

$$\eta(t) \triangleq \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}, \quad \zeta(t) \triangleq \begin{bmatrix} \omega(t) \\ \upsilon(t) \end{bmatrix}$$

$$\bar{A}_0 \triangleq \begin{bmatrix} A_0 & 0 \\ A_0 - G - KC_0 & G \end{bmatrix}, \quad \bar{A}_1 \triangleq \begin{bmatrix} A_1 & 0 \\ A_1 - KC_1 & 0 \end{bmatrix}$$

$$B \triangleq \begin{bmatrix} B_1 & 0 \\ B_1 & -KC_2 \end{bmatrix}, \quad C \triangleq \begin{bmatrix} 0 & I \end{bmatrix}$$

The H_2 norm augmented system G_a is used as the performance index of estimation

$$\|G_a\|_2^2 = J(G,k,h) = \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T e'(t)\, e(t)dt\right\} \tag{6}$$

If there are no time delay terms (i.e., $A_1 = 0$ and $C_1 = 0$), then (1) becomes

$$\dot{x}(t) = A_0 x(t) + B_1\omega(t) + B_2 u(t)$$
$$y(t) = C_0 x(t) + C_2 \nu(t)$$

and the filter, minimizing the H_2 norm (6) for this non-delayed system, is the standard Kalman filter. Thus we can call the proposed filter minimizing (6) a Kalman filter for time delay systems.

3 H$_2$ Norm Computatuon

The H_2 norm of G_a is expressed in terms of matrix function $P(s)$ in the next theorem.

Theorem 1: If is stable, then

$$\|G_a\|_2^2 = Tr(B'P(0)B) \tag{7}$$

Where $P(s), 0 \le s \le h$ is continuously differentiable and satisfies

$$P(0) = P'(0)$$
$$P(s) = \bar{A}_0' P(0) + \bar{A}_1' P(h-s), \, 0 \le s \le h \tag{8}$$
$$\dot{P}(0) + \dot{P}'(0) + C'C = 0$$

Remark 1: is related to the Lyapunov functional of state delay system (4). Let $V(\phi), \phi \in C[-h,0]$ be defined by

$$V(\phi) \triangleq \phi'(0)P(0)\phi(0) + 2\phi'(0)\int_0^h P(r)\bar{A}_1\phi(-h+r)dr + \int_0^h \phi'(-h+r)\int_0^h \bar{A}_1' P(r-s)\bar{A}_1\phi(-h+r)dsdr \tag{9}$$

Where $P(s) \triangleq P'(-s)$ if $s < 0$. Equation (8) is derived from

$$\frac{d}{dt}V(x_t) = -x'(t)x(t) \tag{10}$$

Where

$$x_t(r) \triangleq x(t+r), \quad r \in [-h, 0]$$

Remark 2: If there are no time delay terms, the result in Theorem 1 becomes a standard H_2 norm computation. See, for example, Theorem 3.3.1 in [25]: the H_2 norm of a stable non-delay system is given by

$$\|G_a\|_2^2 = Tr(B'PB) \tag{11}$$

Where

$$\bar{A}_0' P + P\bar{A}_0 + C'C = 0$$

Note that conditions (7) are equivalent to those in (11) if $h = 0$.
The proof of Theorem 1 will be given using Lemma 1 and 2.

Lemma 1: If system G_a is stable, then

$$\|G_a\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Tr(G_a(j\omega)G_a'(-j\omega))d\omega \tag{12}$$

Proof: The result is standard (see Chap 3.3 in [25]).
Lemma 2: If G_a is stable and $P(s), 0 \le s \le h$ satisfies (8), then

$$P(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Delta^{-1}(j\omega)'\Delta^{-1}(-j\omega)d\omega \tag{13}$$

Where

$$\Delta(j\omega) \triangleq j\omega I - \bar{A}_0 - \bar{A}_1 e^{-j\omega h} \tag{14}$$

Proof: See [26].
(Proof of Theorem 1) From Lemma 1,

$$Tr(B'P(0)B)$$
$$= Tr\left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty} B'\Delta^{-1}(j\omega)'\Delta^{-1}(-j\omega)Bd\omega\right\}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} Tr\left\{B'\Delta^{-1}(j\omega)'\Delta^{-1}(-j\omega)B\right\}d\omega$$

Since $\int_{-\infty}^{+\infty} f(j\omega)d\omega = \int_{-\infty}^{+\infty} f(-j\omega)d\omega$, we have

$$
\begin{aligned}
& Tr(B'P(0)B) \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} Tr\left\{B'\Delta^{-1}(-j\omega)'\Delta^{-1}(j\omega)B\right\}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} Tr\left\{G_a'(-j\omega)'G_a(j\omega)\right\}d\omega
\end{aligned}
$$

Since $Tr(AB)=Tr(BA)$ whenever AB and BA are square matrices, we have

$$Tr(B'P(0)B) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Tr\left\{G_a(j\omega)'G_a'(-j\omega)\right\}d\omega = \left\|G_a\right\|_2^2$$

The last equality is from (12).
If G_a is stable, then $\left\|G_a\right\|_2^2$ can be computed from $P(0)$ in Theorem 1. How to check the stability of G_a will be considered later in Theorem 2; first we will consider how to compute $P(0)$ in the next lemma.

Notation: For a matrix $M \in \mathbb{C}^{n\times n}$ given by

$$M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}$$

M' denotes complex conjugate transpose of M
the column string csM is defined by

$$csM \triangleq \left[m_{11}\ m_{12}\cdots m_{1n}\ |\ m_{21}\ m_{22}\cdots m_{2n}\ |\ \cdots |\ m_{n1}\ m_{n2}\cdots m_{nn}\right]' \in \mathbb{C}^{n^2\times 1}$$

How to compute $P(s), 0 \le s \le h$, is considered in the next lemma.

Lemma 3: If G_a is stable, then $P(0)$ and $P(h)$ satisfying (8) are given by

$$\begin{bmatrix} (I\otimes\bar{A}_0') + (\bar{A}_0'\otimes I) & (I\otimes\bar{A}_1')T + (\bar{A}_1'\otimes I) \\ R_1 & R_2 \end{bmatrix} \cdot \begin{bmatrix} csP(0) \\ csP(h) \end{bmatrix} = \begin{bmatrix} -csC'C \\ 0 \end{bmatrix} \quad (15)$$

Where

$$T := \left[T_1\,|T_2\,|\cdots|T_{n^2}\right], T_k \in R^{n^2\times 1}$$

Row vector $T_k, 1\le k \le n^2$ is defined by

$$T_k, 1\le k \le n^2\ T_{(i-1)n+j} := e_{(j-1)n+i}\ , 1\le i, j \le n^2$$

Where $e_k \in R^{n^2\times 1}, 1\le k \le n^2$ is a row vector whose k-th element is 1 and all other elements are 0.
And

$$\left[R_1 \quad R_2\right] \triangleq \left[\Sigma_1 \quad 0\right]V^*$$

Matrices Σ_1 and V^* are from the singular value decomposition of the following

$$\left(I - J\exp(Hh)\right) = U\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}V^* \quad (16)$$

Where U and V are unitary matrices, and $\Sigma_1 \in R^{n^2\times n^2}$ is a diagonal matrix whose diagonal elements are nonzero singular values of $\left(I - J\exp(Hh)\right)$. Let T_{ij} denote an $n\times n$ matrix with (i,j)-entry equal to 1 and all other entries equal to zero, and let $T \in R^{n^2\times n^2}$ be the block matrix T , $\left[T_{ji}\right]$ (i.e., the (i,j)-block of T is T_{ji}). Matrices H and J are defined by

$$H \triangleq \begin{bmatrix} (I\otimes\bar{A}_0') & (I\otimes\bar{A}_1')T \\ -(I\otimes\bar{A}_1')T & -(I\otimes\bar{A}_0') \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

Proof: See [22].
Note that $P(0)$ can be computed from the matrix exponential (16) and a simple linear equation (15). Thus if G_a is stable, then we can easily compute H_2 norm: see (7).
Now the stability of G_e is considered in Theorem 2, where a stability condition for interval delay $h \in \left[0, \bar{h}\right)$ is provided.

Theorem 2: Suppose G_a is stable for $h = 0$. If H has imaginary eigenvalues $\left\{j\omega_1,\ \cdots\ , j\omega_k\right\}$ and their corresponding eigenvectors are given by

$$v_1 = \begin{bmatrix} v_{1,1} \\ v_{1,2} \\ \vdots \\ v_{1,2n^2} \end{bmatrix}, \cdots, v_k = \begin{bmatrix} v_{k,1} \\ v_{k,2} \\ \vdots \\ v_{k,2n^2} \end{bmatrix}$$

then G_a is stable for $h \in \left[0, \bar{h}\right)$ where $\bar{h}$ is defined by

$$\bar{h} \in \min_{1 \le i \le k} \left| \frac{1}{j\omega} \ln\left(\frac{v_{i,l}}{v_{i,l+n^2}} \right) \right| \quad (17)$$

where $v_{i,l}, 0 \le 1 \le n^2$ is any nonzero element of v_l. Theorem 2 is proved using Lemma 4 and 5. Lemma 4 is based on the fact that if G_a is stable for $h = 0$ and G_a does not have any imaginary poles for $h \in \left[0, \bar{h}\right)$, then G_a is stable for $h \in \left[0, \bar{h}\right)$.

Lemma 4: G_a is stable for $h \in \left[0, \bar{h}\right)$ if

- G_a is stable for $h = 0$
- The following equation does not have any roots for $h \in \left[0, \bar{h}\right)$:

$$\det(j\omega I - \bar{A}_0 - \bar{A}_1 e^{-j\omega h}) = 0 \quad (18)$$

Proof: See [27].

Stability of G_a for $h = 0$ can be easily checked from eigenvalues of $\bar{A}_0 + \bar{A}_1$. On the other hand, checking whether (18) has any roots for $h \in \left[0, \bar{h}\right)$ is not easy: (18) should be checked for all $0 \le \omega < \infty$ and $0 \le h < \bar{h}$ In the next lemma, it is shown that a root $j\omega$ of (18) (if any) is an eigenvalue of H.

Lemma 5: If (18) has a root ω, then it is an eigenvalue of H.

Proof: Suppose (18) has a root $j\omega$ for h; then there exists $x\,(\in C^n) \ne 0$ such that

$$x'(j\omega I - \bar{A}_0 - \bar{A}_1 e^{-j\omega h}) = 0 \quad (19)$$

Taking the transpose (not complex conjugate), we Obtain

$$(j\omega I - \bar{A}_0 - \bar{A}_1 e^{-j\omega h})x = 0$$

Let $\alpha \in C^n$ be defined by

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \triangleq x e^{-\frac{j\omega h}{2}} \quad (20)$$

where , $\alpha_i, 1 \le i \le n$ is a complex number. Let v be defined by ($\bar{u}$ is the complex conjugate of u)

$$v \triangleq \begin{bmatrix} u \\ \bar{u} \end{bmatrix} \quad (21)$$

Where

$$u = \begin{bmatrix} \bar{\alpha}_1 x \\ \bar{\alpha}_2 x \\ \vdots \\ \bar{\alpha}_n x \end{bmatrix} \in C^{n^2} \quad (22)$$

The theorem is proved if we show that this v ($v \ne 0$ from the construction) satisfies $(j\omega I - H)v = 0$: that is, $j\omega$ is an eigenvalue of H. From the definition of H, we obtain

$$\begin{aligned} &(j\omega I - H)v \\ &= \begin{bmatrix} j\omega I - (I \otimes \bar{A}_0') & -(I \otimes \bar{A}_1')T \\ (I \otimes \bar{A}_1')T & j\omega I + (I \otimes \bar{A}_0') \end{bmatrix} v \\ &= \begin{bmatrix} \left(j\omega I - (I \otimes \bar{A}_0')\right)u - (I \otimes \bar{A}_1')T\bar{u} \\ \left(j\omega I + (I \otimes \bar{A}_0')\right)\bar{u} + (I \otimes \bar{A}_1')Tu \end{bmatrix} \end{aligned} \quad (23)$$

Partition $(j\omega I - H)v$ into $2n$ complex vectors and let the i-th block of $(j\omega I - H)v$ be denoted by $r_i \in C^n$. Then $r_i, 1 \le i \le n$ is given by

$$r_i = (j\omega I - \bar{A}_0')\bar{\alpha}_i x - \bar{A}_1'(T_{1i}\bar{\alpha}_1 + T_{2i}\bar{\alpha}_2 + \cdots + T_{ni}\bar{\alpha}_n)\bar{x}$$

Noting the following relation

$$\begin{aligned} &(T_{1i}\bar{\alpha}_1 + T_{2i}\bar{\alpha}_2 + \cdots + T_{ni}\bar{\alpha}_n)\bar{x} \\ &= (T_{1i}\bar{\alpha}_1 + T_{2i}\bar{\alpha}_2 + \cdots + T_{ni}\bar{\alpha}_n)\bar{\alpha} e^{-\frac{j\omega h}{2}} \\ &= e^{-\frac{j\omega h}{2}} \bar{\alpha}_i \alpha \end{aligned}$$

We obtain

$$\begin{aligned} r_i &= (j\omega I - \bar{A}_0')\bar{\alpha}_i \alpha e^{\frac{j\omega h}{2}} - \bar{\alpha}_i \bar{A}_1' \alpha e^{-\frac{j\omega h}{2}} \\ &= \bar{\alpha}_i e^{\frac{j\omega h}{2}} \left(j\omega I - \bar{A}_0' - \bar{A}_1' e^{-j\omega h} \right) \alpha \\ &= \bar{\alpha}_i \left(j\omega I - \bar{A}_0' - \bar{A}_1' e^{-j\omega h} \right) x = 0, \;\; 1 \le i \le n \end{aligned}$$

The last equality is from (19).

Since $r_{i+n} = -\bar{r}_i$, $1 \le i \le n$ (see (23)), we have $r_i = 0,\ n+1 \le i \le 2n$. Hence , $(j\omega I - H)v = 0$, where $v \ne 0$ (since $x \ne 0$).

Proof of Theorem 2: From the proof of Lemma 5, if (18) has a root ω_i for h_i $(1 \le i \le k)$, then ω_i is an eigenvalue of H. Furthermore, the corresponding eigenvector of H is of the form:

$$v_i = \left[\bar{x}_1 x e^{\frac{j\omega_i h_i}{2}} \quad \bar{x}_2 x e^{\frac{j\omega_i h_i}{2}} \cdots \bar{x}_n x e^{\frac{j\omega_i h_i}{2}} \quad x_1 \bar{x} e^{-\frac{j\omega_i h_i}{2}} \quad x_2 \bar{x} e^{-\frac{j\omega_i h_i}{2}} \cdots x_n \bar{x} e^{-\frac{j\omega_i h_i}{2}} \right]^T$$

Thus h_i can be computed as follows:

$$h_i = \left| \frac{1}{j\omega} \ln \left(\frac{v_{i,l}}{v_{i,l+n^2}} \right) \right|$$

Where $v_{i,l}$, $1 \le l \le n^2$ is any nonzero element of v_i. If the minimum value of h_i $(1 \le i \le k)$ is h_i then (18) does not have a root for $h \in \left[0, \bar{h}\right)$. From Lemma 4, this proves the theorem.

Remark 3: Once (G, K) is determined, we can check the stability of the error system (4) (Theorem 2) and compute its H_2 norm (Theorem1).

4 Kalman Filter for Time Delay Systems: Synthesis

In this section, the synthesis algorithm of Kalman filter (3) is proposed, where the algorithm is formulated as a constrained nonlinear optimization problem and the output delay $h = h^*$.

When minimizing H_2 norm of G_a over (G, K) using Theorem 1, it should be guaranteed that G_a is stable. The approach presented here allows one to design linear observers for time delay systems (see Fig.1). If (G, K) is given, the stability of G_a can be checked using Theorem 2, which provides a upper stability bound $\bar{h}(k)$ (i.e., G_a is stable as long as $h < \bar{h}$). Thus finding (G, K), which stabilizes G_a and minimizes $\|G_a(G, K, h)\|_2$.

Kalman filter design problem can be formulated as follows:

$$\begin{aligned} &\min_{G,k} J_1(G, K, h^{/*}) \triangleq \left\| G_a(G, K, h^*) \right\|_2^2 \\ &\textit{subject to } h < \bar{h}(G, K) \end{aligned} \tag{24}$$

(24) is a constrained nonlinear optimization problem whose global solution is difficult to find. A suboptimal approach is proposed to compute (G, K) using penalty methods [26].

A penalty function is defined by

$$p(G, K) \triangleq \begin{cases} 0 & \textit{if} \quad h < \bar{h}(G, K) \\ \alpha (h^* - \bar{h})^2 & \textit{if} \quad h \ge \bar{h}(G, K) \end{cases}$$

where α is a constant and is chosen so that $p(G, K, h^*) \gg J(G, K, h^*)$ when $h^* \gg \bar{h}(G, K)$. With this penalty function, a constrained optimization problem (24) can be replaced by the following unconstrained optimization problem:

$$\min_{G,K} J_2(G, K, h^*) \triangleq \left\| G_a(G, K, h^*) \right\|_2^2 + p(G, K) \tag{25}$$

Note that if $h^* < \bar{h}(G, K)$ (i.e., G_a is stable), then $J_2(G, K, h^*) = J_1(G, K, h^*)$. Also note that if $h^* \ge \bar{h}(G, K)$,then $J_2(G, K, h^*)$ is dominated by the penalty function $p(G, K, h^*)$. Thus the penalty function $p(G, K, h^*)$ prevents unstable region searching when the H_2 norm is being minimized.

initial value of G and K can be chosen by minimizing $J(G, K, 0)$: the initial value corresponds to the Kalman filter gain for a non-delayed system. Minimization problem (25) can be solved, for example, using an unconstrained nonlinear optimization function fminunc in MATLAB optimization toolbox.

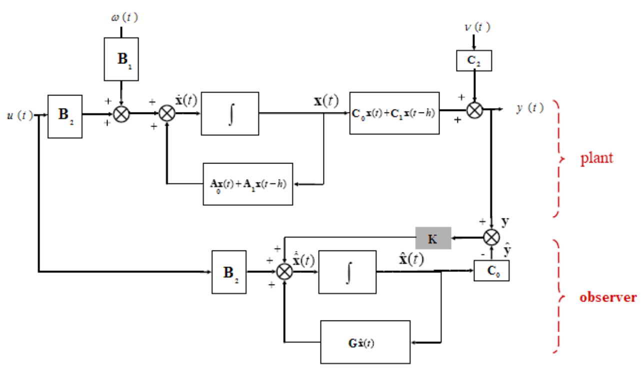

Fig. 1: The block diagram of observer

5 Numerical Example

In this section, the simulations have been performed by means of the MATLAB software.

Example 1: Consider the following first-order time delay system:

$$\begin{aligned} \dot{x}(t) &= -x(t) - 2x(t-h) + 0.5\omega(t) + u(t) \\ y(t) &= x(t) + x(t-h) + 0.5v(t) \end{aligned} \tag{26}$$

where $\omega(t)$ and $v(t)$ are the vectors of the input noise and measurement noise, respectively. It is assumed that these noises are Gaussian processes with an average of zero and that $\omega(t)$ and $v(t)$ are uncorrelated and they satisfy relation (2). In this example, $h = 0.5$.

The optimization problem (25) is solved by means of the Matlab optimization toolbox, and for this purpose, the optimization function "fminunc" in Matlab is used.

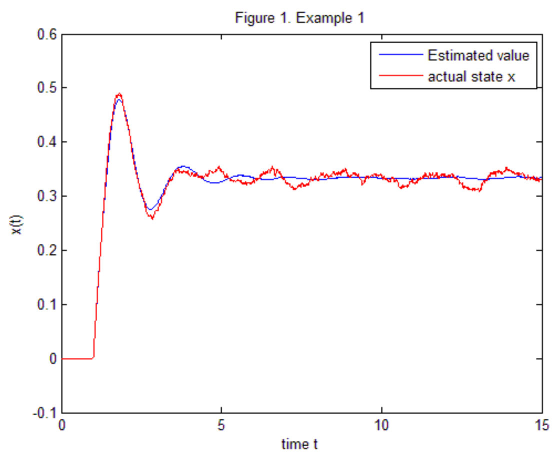

Fig. 2: Simulation result : true state and estimated value

By using $h = 0$, the initial value for (G, K) is obtained. The value of α in the penalty function has been adjusted at 200. The values calculated for $h = 0.5$ are as follows:

$$\|G_e(K,h)\|_2^2 = 0.0717$$

Using the computed filter gain, state estimation simulation was done, where a unit step signal was applied to the control input $u(t)$ at time 1s. The simulation result is given in Fig.2

it can be seen that the proposed Kalman filter estimates system states well.

To see how the time delay affects estimation performance, Kalman filters were designed for different h values.

As seen in Table 1, computed H_2 norm increases as time delay h increases.

Table 1. Time delay effects on estimation performance.

	$h=0.1$	$h=0.3$	$h=0.5$	$h=0.8$
$\|G_e(K,h)\|_2^2$	0.0399	0.0543	0.0717	0.1095
Variance of actual estimation error	0.000015	0.00035	0.00065	0.0009

Example 2: In this problem, the H_2 filter is designed for the second-order system given in the following relation.

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-h) \\ &\quad + \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} \omega(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 1 \end{bmatrix} x(t-h) + 0.5v(t) \end{aligned} \tag{27}$$

where $\omega(t)$ and $v(t)$ are zero-mean, uncorrelated white Gaussian processes satisfying (2). The time delay is set to be $h = 0.3$.

Optimization problem (25) was solved using Matlab optimization toolbox. The initial value of (G, K) is computed using $h = 0$, and α in the penalty function is set to 100. The computed values are as follows:

$$\bar{h} = 1.6309, \|G_e(K,h)\|_2^2 = 0.0240$$

Using the computed (G, K), state estimation simulation was done, where a unit step signal was applied to the control input $u(t)$ at time 1s. The simulation results are given in Fig.3 and Fig.4: it can be seen that the proposed Kalman filter estimates system states well.

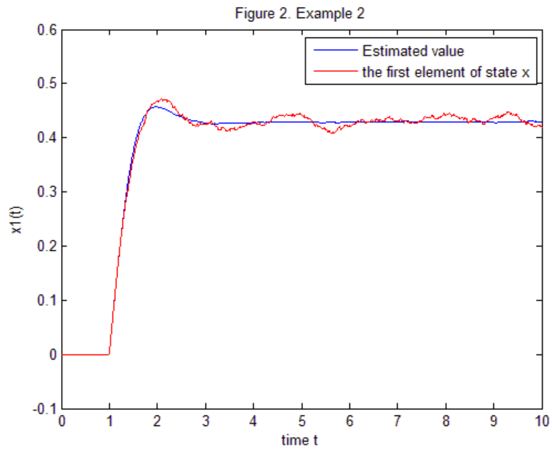

Fig. 3: Simulation result: true state (the first element of state x) and estimated value

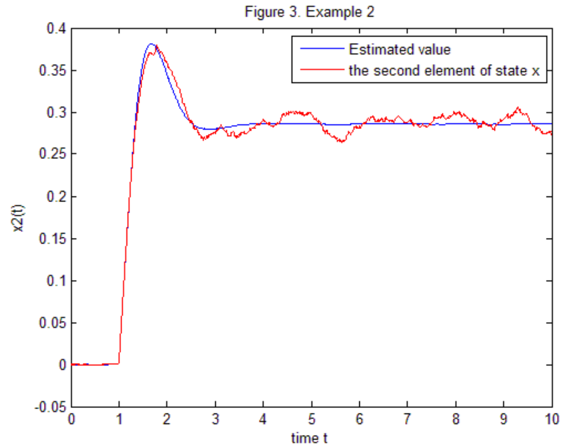

Fig. 4: Simulation result: true state (the second element of state x) and estimated value

To see how the time delay affects estimation performance, Kalman filters were designed for different h values.

As seen in Table 2, computed H_2 norm increases as time delay h increases. Variance of actual estimation error, which was computed from a simulation, also increases as time delay h increases. This verifies a common belief that the time delay adversely affects on estimation performance.

Table 2. Time delay effects on estimation performance.

	$h=0.1$	$h=0.3$	$h=0.5$	$h=0.7$
$\lVert G_e(K,h)\rVert_2^2$	0.0180	0.0243	0.0321	0.0424
Variance of actual estimation error	0.00088	0.00011	0.00013	0.00015

Example 3: Consider the following third-order system with delayed output and state:

$$\dot{x}(t)=\begin{bmatrix}-1 & 13.5 & -1\\ -3 & -1 & -2\\ -2 & -1 & -4\end{bmatrix}x(t)+\begin{bmatrix}-5.9 & 7.1 & -70.3\\ 2 & -1 & 5\\ 2 & 0 & 6\end{bmatrix}x(t-h)+\begin{bmatrix}0.2\\ 0.2\\ 0.2\end{bmatrix}\omega(t)+\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}0 & 0 & 1\end{bmatrix}x(t)+\begin{bmatrix}1 & 1 & 1\end{bmatrix}x(t-h)+0.5v(t) \tag{28}$$

where $\omega(t)$ and $v(t)$ are the vectors of the input noise and measurement noise, respectively. In this example $h=0.06$.

The optimization problem (25) is solved by means of the Matlab optimization toolbox, and for this purpose, the optimization function "fminsearch" in Matlab is used.

By using $h=0$, the initial value for (G,K) is obtained. The value of α in the penalty function has been adjusted at 50. The values calculated for $h=0.06$ are as follows:

$$\bar{h}=0.1624,\ \lVert G_e(K,h)\rVert_2^2=1.3949$$

The simulation results are given in Fig.5, Fig.6 and Fig.7: it can be seen that the proposed H_2 filter estimates system states well.

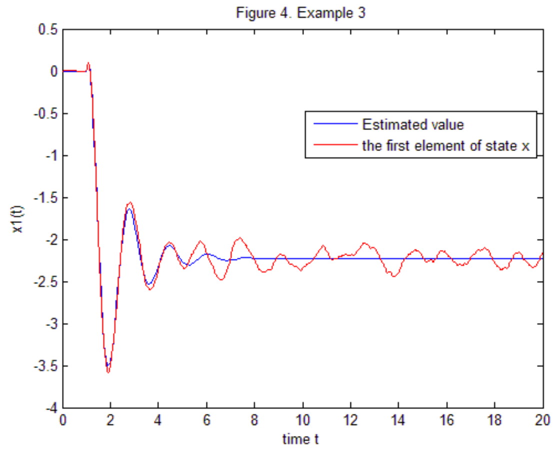

Fig. 5: Simulation result: true state (the first element of state x) and estimated value

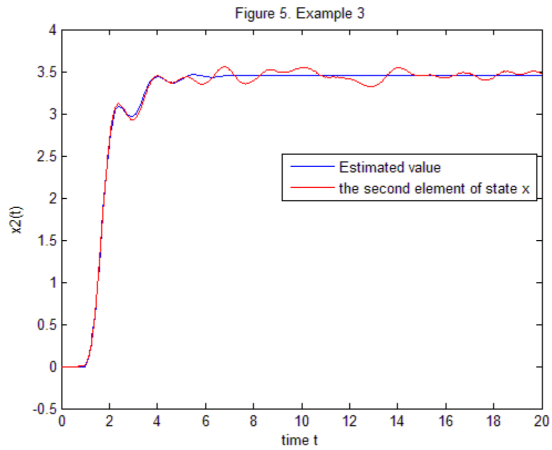

Fig. 6: Simulation result: true state (the second element of state x) and estimated value

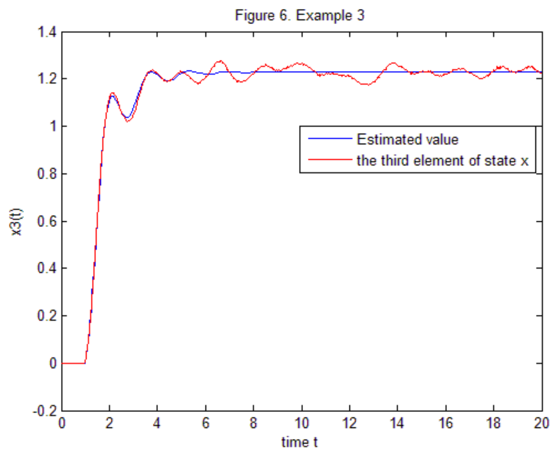

Fig. 7: Simulation result: true state (the third element of state x) and estimated value

As seen in Table 3, computed H_2 norm increases as time delay h increases.

Table 3. Time delay effects on estimation performance.

	$h = 0.01$	$h = 0.03$	$h = 0.06$	$h = 0.1$
$\|G_e(K,h)\|_2^2$	0.9765	1.0962	1.3949	2.0859
Variance of actual estimation error	0.000055	0.0001	0.0005	0.0007

As is observed, the increase of time delay has an opposite effect on the estimation performance, and with the increase of time delay, the estimation error variance also increases.

6 Conclusion

In this article, a method was proposed for the designing of Kalman filter for linear systems with time delay in the output and in state variables. By using the finite characterization of a Lyapunov functional equation, the existence of sufficient conditions for achieving the right solution and guaranteeing the proper convergence rate of the estimation error was evaluated. This observer provided satisfactory results in practical applications. Finally, by designing observers for three linear systems with time delays, the effectiveness of the proposed approach was demonstrated.

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