On Ulam's type stability

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Quite often (e.g., in applications) we have to do with functions that satisfy some equations only approximately. There arises a natural question what errors we commit when we replace such functions by the exact solutions to those equations. Some tools to evaluate them are provided within the theory of the Ulam (also Hyers-Ulam) type stability.

The issue of Ulam's type stability of (first, functional, but next also difference, differential and integral) equations has been a very popular subject of investigations for the last nearly fifty years (see, e.g., [3, 8, 9, 10]). The main motivation for it was given by S.M. Ulam in 1940. The following definition somehow describes the main ideas of such stability notion for equations in n variables ($\mathbb{R}_+$ stands for the set of nonnegative reals).

Definition 1. *Let A be a nonempty set, (X, d) be a metric space, $\mathcal{C} \subset {\mathbb{R}_+}^{A^n}$ be nonempty, $\mathcal{T}$ map $\mathcal{C}$ into ${\mathbb{R}_+}^{A}$, and $\mathcal{F}_1, \mathcal{F}_2$ map a nonempty $\mathcal{D} \subset X^A$ into X^{A^n}. We say that the equation*

$$\mathcal{F}_1\varphi(x_1, \dots, x_n) = \mathcal{F}_2\varphi(x_1, \dots, x_n) \tag{1}$$

is $\mathcal{T}$ – stable provided for every $\varepsilon \in \mathcal{C}$ and $\varphi_0 \in \mathcal{D}$ with

$$d\big(\mathcal{F}_1\varphi_0(x_1, \dots, x_n), \mathcal{F}_2\varphi_0(x_1, \dots, x_n)\big) \le \varepsilon(x_1, \dots, x_n), \qquad x_1, \dots, x_n \in A,$$

there is a solution $\varphi \in \mathcal{D}$ of equation (1) such that $d\big(\varphi(x), \varphi_0(x)\big) \le \mathcal{T}\varepsilon(x)$ for $x \in A$.

The next two theorems contain examples of some results on stability of the additive Cauchy equation (see [3]) and of a linear difference equation of higher order (see [7]).

Theorem 1. *Let E_1 and E_2 be two normed spaces, $c \ge 0$ and $p \ne 1$ be fixed real numbers. Let $f : E_1 \to E_2$ be such that*

$$\|f(x+y) - f(x) - f(y)\| \le c(\|x\|^p + \|y\|^p), \qquad x, y \in E_1 \setminus \{0\}.$$

If $p < 0$, then f is additive (i.e., $f(x+y) = f(x) + f(y)$ for $x, y \in E_1$). If $p \ge 0$ and E_2 is complete, then there is a unique additive $T : E_1 \to E_2$ with

$$\|f(x) - T(x)\| \le \frac{c\|x\|^p}{\left|2^{p-1} - 1\right|}, \qquad x \in E_1 \setminus \{0\}.$$

Theorem 2. *Let T be either $\mathbb{N}$ or $\mathbb{Z}$, X be a Banach space over $F \in \{\mathbb{R}, \mathbb{C}\}$, $(b_n)_{n\in T}$ be a sequence in X, $a_1, \dots, a_m \in F$, $\delta > 0$ and $r_1, \dots, r_m \in \mathbb{C}$ be the roots of the characteristic equation of the difference equation*

$$x_{n+m} = a_1 x_{n+m-1} + \dots + a_m x_n + b_n, \qquad n \in T. \tag{2}$$

Suppose that $|r_i| \ne 1$ for $i = 1, \dots, m$ and $(y_n)_{n\in T}$ is a sequence in X with

$$\|y_{n+m} - a_1 y_{n+m-1} - \dots - a_m y_n - b_n\| \le \delta, \qquad n \in T.$$

Then there exists a sequence $(x_n)_{n\in T}$ in X such that (2) holds and

$$\|y_n - x_n\| \le \frac{\delta}{|1 - |r_1|| \cdot \ldots \cdot |1 - |r_m||}, \qquad n \in T.$$

The lecture contains some basic motivations, definitions and results connected with the notion of the Ulam (but also the Hyers-Ulam) type stability. A general method will also be presented for investigations of that stability, e.g., of the following linear (difference, differential, functional) equations of higher orders:

$$b_m\varphi(n+m) + b_{m-1}\varphi(n+m-1) + \dots + b_1\varphi(n+1) + b_0\varphi(n) = G(n),$$

$$b_m\varphi^{(m)}(z)+b_{m-1}\varphi^{(m-1)}(z)+\cdots+b_1\varphi'(z)+b_0\varphi(z)=G(z),$$
$$b_m\varphi(f^m(z))+b_{m-1}\varphi(f^{m-1}(z))+\cdots+b_1\varphi(f(z))+b_0\varphi(z)=G(z).$$

It works for analogous integral equations, as well. In many cases, functions satisfying such equations approximately generate the exact solutions to them (see, e.g., [2]). That method can be described in the terms of fixed points in suitable function spaces (for related results see, e.g., [1, 5, 6]). Some examples of simple applications of it are provided.

References

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Janusz Brzdęk (short biography)

Present permanent employment: Department of Mathematics, Pedagogical University, Kraków, Poland; position of professor

1983 – Master of Science in Mathematics, Jagiellonian University, Kraków, Poland

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Lead Editor of Banach Center Publications volume 99 (2013) titled: *Recent Developments in Functional Equations and Inequalities. Selected Topics*

Lead Guest Editor of Abstract and Applied Analysis annual special issues: *Ulam's Type Stability* (http://www.hindawi.com/journals/aaa/type.stability/) in the years 2012, 2013, 2014

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