On some selected issue in stability of functional equations

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There arises a natural question what errors we commit when we replace functions that satisfy some equations only approximately by the exact solutions to those equations. Some tools to evaluate them are provided within the theory of stability of functional equations (nowadays often called Ulam's type stability), which has been developed in connection with a question posed by S.M. Ulam in 1940 (see [3, 7, 14, 15, 16]). The talk contains some basic definitions and (early and recent) results concerning that stability of functional equations; in particular, the notions of superstability and hyperstability are discussed. The following three definitions describe partially the main ideas of them (see [7]).

Definition 1. *Let A be a nonempty set, (X, d) be a metric space, $\mathcal{E} \subset \mathcal{C} \subset \mathbb{R}_+{}^{A^n}$ be nonempty, $\mathcal{T}$ be an operator mapping $\mathcal{C}$ into $\mathbb{R}_+{}^{A}$ and $\mathcal{F}_1, \mathcal{F}_2$ be operators mapping a nonempty set $\mathcal{D} \subset X^A$ into X^{A^n}. We say that the equation*

$$\mathcal{F}_1\varphi(x_1, \ldots, x_n) = \mathcal{F}_2\varphi(x_1, \ldots, x_n) \tag{1}$$

is $(\mathcal{E}, \mathcal{T})$ – stable provided for any $\varepsilon \in \mathcal{E}$ and $\varphi_0 \in \mathcal{D}$ with

$$d\big(\mathcal{F}_1\varphi_0(x_1, \ldots, x_n), \mathcal{F}_2\varphi_0(x_1, \ldots, x_n)\big) \le \varepsilon(x_1, \ldots, x_n), \qquad x_1, \ldots, x_n \in A, \tag{2}$$

there exists a solution $\varphi \in \mathcal{D}$ of equation (1) such that

$$d\big(\varphi(x), \varphi_0(x)\big) \le \mathcal{T}\varepsilon(x), \qquad x \in A. \tag{3}$$

Roughly speaking, $(\mathcal{E}, \mathcal{T})$ – stability of equation (1) means that every approximate (in the sense of (2)) solution of (1) is always close (in the sense of (3)) to an exact solution of (1).

Definition 2. *Let A be a nonempty set, (X, d) be a metric space, $\varepsilon \in \mathbb{R}_+{}^{A^n}$ and $\mathcal{F}_1, \mathcal{F}_2$ be operators mapping a nonempty set $\mathcal{D} \subset X^A$ into X^{A^n}. We say that equation (1) is ε – hyperstable provided every $\varphi_0 \in \mathcal{D}$, satisfying (2), fulfils equation (1).*

Definition 3. *Let A be a nonempty set, (X, d) be a metric space and $\mathcal{F}_1, \mathcal{F}_2$ be operators mapping a nonempty set $\mathcal{D} \subset X^A$ into X^{A^n}. We say that operator equation (1) is superstable if every $\varphi \in \mathcal{D}$, that is unbounded (i.e., $\sup_{x,y \in A} d(\varphi(x), \varphi(y)) = \infty$) and satisfies*

$$\sup_{x_1, \ldots, x_n \in A} d\big(\mathcal{F}_1\varphi(x_1, \ldots, x_n), \mathcal{F}_2\varphi(x_1, \ldots, x_n)\big) < \infty,$$

is a solution of equation (1).

The lecture focuses, in particular, on stability of the difference equation of the form

$$x_{n+1} = F(x_n),$$

its generalizations and related functional equations. Also, stability of some conditional equations of the forms

$$g(x + y) = g(x) + g(y), \qquad x \perp y,$$
$$g(x + y) = g(x)g(y), \qquad x \perp y,$$

are considered (cf. [4, 5, 6]) with some relations $\perp$ patterned on some classical orthogonality notions.

Some examples of stability outcomes for differential and integral equations (cf. [1, 2]) are provided, as well.

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Janusz Brzdęk (short biography)

Present permanent employment: Department of Mathematics, Pedagogical University, Kraków, Poland; position of professor

1983 – Master of Science in Mathematics, Jagiellonian University, Kraków, Poland

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Lead Editor of Banach Center Publications volume 99 (2013) titled: *Recent Developments in Functional Equations and Inequalities. Selected Topics*

Lead Guest Editor of Abstract and Applied Analysis annual special issues: *Ulam's Type Stability* (http://www.hindawi.com/journals/aaa/type.stability/) in the years 2012, 2013

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