Robust Extended Recursive Wiener Fixed-Point Smoother and Filter in Discrete-Time Stochastic Systems

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Abstract: - As an extension of the linear robust recursive least-squares Wiener fixed-point smoother and filter, this paper originally designs the robust extended recursive Wiener fixed-point smoother and filter for estimating the signal in discrete-time wide-sense stationary stochastic systems. It is a characteristic in this paper that the signal is modulated with the nonlinear mechanism. As a step to the estimation problem for the observation mechanism with the nonlinear modulation, the robust signal estimators are proposed for the observation equation with the linear amplitude modulation of the signal. The observation noise is additional white noise. The system matrix in the state equation contains uncertain parameters. The robust extended recursive Wiener estimators are derived from the Wiener-Hopf equation. In the simulation example, it is shown that the proposed robust extended recursive Wiener fixed-point smoother and filter are superior in estimation accuracy to the extended recursive Wiener estimators.

Key-Words: - Discrete-time stochastic systems, robust extended recursive Wiener estimators, covariance information, fixed-point smoother, nonlinear modulation

1 Introduction

In [1], the extended recursive Wiener fixed-point smoother and filter are presented in discrete-time stochastic systems with the nonlinear observation mechanism of the signal. Fang et. al. [2] introduce the Bayesian state estimation framework and review various techniques, from the standard Kalman filter for linear systems to extended Kalman filter, unscented Kalman filter and ensemble Kalman filter for nonlinear stochastic systems. Bayesian estimation methods include the Gaussian filtering, Gaussian-sum filtering, particle filtering and moving horizon estimation. The discussion of state estimation is extended to more complicated problems such as simultaneous state and parameter/input estimations. In [3], two algorithms of the extended unbiased finite impulse response (FIR) filtering of nonlinear discrete-time state-space models are discussed. Unlike the extended Kalman filter, both extended FIR algorithms demonstrate better robustness against model uncertainties. In [4], the robust Kalman filter is designed for systems involving unknown parameter perturbations with norm-bounded uncertainties. In [5], the robust recursive least-squares (RLS) Wiener fixed-point smoother and filter are designed in the signal estimation problem for the linear discrete-time stochastic systems with uncertain parameters in the system and observation matrices. In [6], the robust RLS Wiener FIR filter is proposed in linear discrete-time stochastic systems with uncertain parameters in the system and observation matrices. In [7], the robust extended Kalman filter is designed in discrete-time stochastic systems. The algorithm is applied to the pulsar positioning system. In [8], the robust filter is designed for discrete time nonlinear systems including uncertainties. The nonlinear functions are assumed to be uncertain but belonging to a conic region. The design method also allows dynamic and measurement noises having unknown time-varying expected values, covariances and cross-covariances. In [9], the robust extended Kalman filter is designed to estimate the rotor angles and the rotor speeds of synchronous generators of a multi-machine power system.

As an extension of the linear robust RLS Wiener fixed-point smoother and filter [5] in linear discrete-time stochastic systems, this paper, in Theorem 2, originally proposes the robust extended recursive Wiener fixed-point smoothing and filtering algorithms for estimating the signal for discrete-time wide-sense stationary stochastic systems. As a first step to the robust extended recursive Wiener estimators, Theorem 1 proposes the robust RLS Wiener fixed-point smoothing and filtering algorithms for estimating the signal in the stochastic systems with the linear amplitude modulation of the observation mechanism. It is assumed that the system matrix in the state equation contains uncertain parameters. It is a characteristic in this paper that the signal is modulated with the nonlinear mechanism of the signal. The observation noise is additional white noise. The robust extended recursive Wiener estimators are derived based on the robust RLS Wiener estimators in Theorem 1.

In the simulation example, the phase demodulation of the signal is dealt with. The phase demodulation from the phase modulated signal is important in the analog and digital communication systems [10]. The estimation accuracy of the proposed robust extended recursive Wiener estimators is compared with the extended recursive Wiener estimators [1].

2 Robust least-squares fixed-point smoothing problem for linear amplitude modulation of signal

Let the state-space model in linear discrete-time stochastic systems be described by

$$\begin{aligned} &y(k) = H(k)z(k) + v(k), z(k) = Cx(k), \\ &x(k+1) = \Phi x(k) + \Gamma w(k), \\ &E[v(k)v(s)] = R\delta_K(k-s), \\ &E[w(k)w^T(s)] = Q\delta_K(k-s). \end{aligned} \tag{1}$$

Here, $z(k)$ represents the scalar signal to be estimated. $H(k)$ is the linear amplitude modulating function for $z(k)$ and $x(k)$ is an $n \times 1$ state vector with the wide-sense stationarity. C is a $1 \times n$ observation vector transforming $x(k)$ to the signal $z(k)$. $v(k)$ is the additional white observation noise. Also, Φ denotes the state transition matrix in the state equation and $w(k)$ is the white noise input. It is assumed that the signal and the observation noise are mutually independent and have zero means. Γ is the n by l input matrix. The auto-covariance functions of the observation noise and the input noise are shown in (1). Let the signal process be expressed by the autoregressive (AR) model of the finite order M.

$$\begin{aligned} &z(k) = -\underline{a}_1 z(k-1) - \underline{a}_2 z(k-2) \cdots \\ &-\underline{a}_M z(k-M) + \underline{e}(k), \\ &E[\underline{e}(k)\underline{e}(s)] = \underline{Q}\delta_K(k-s) \end{aligned} \tag{2}$$

It is assumed that the system matrix Φ in (1) has the general form and is not necessarily limited to the controllable canonical form. For the signal process expressed by the AR model of (2), let the signal $z(k)$ be expressed in terms of the newly introduced state vector $\underline{x}(k)$ as follows.

$$\begin{aligned} &z(k) = \underline{H}\underline{x}(k), \underline{H} = [1 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0], \\ &\underline{x}(k) = \begin{bmatrix} \underline{x}_1(k) \\ \underline{x}_2(k) \\ \vdots \\ \underline{x}_{M-1}(k) \\ \underline{x}_M(k) \end{bmatrix} = \begin{bmatrix} z(k) \\ z(k+1) \\ \vdots \\ z(k+M-2) \\ z(k+M-1) \end{bmatrix} \end{aligned} \tag{3}$$

Then the state equation, corresponding to the AR model (2), is described by

$$\begin{aligned} &\underline{x}(k+1) = \underline{\Phi}\underline{x}(k) + \underline{\Gamma}\underline{w}(k), \\ &E[\underline{w}(k)\underline{w}^T(s)] = \underline{Q}\delta_K(k-s), \\ &\underline{\Phi} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\underline{a}_M & -\underline{a}_{M-1} & -\underline{a}_{M-2} & \cdots & -\underline{a}_1 \end{bmatrix}, \\ &\underline{w}(k) = \underline{e}(k+N). \end{aligned} \tag{4}$$

The system matrix Φ in (4) has the controllable canonical form. By introducing the auto-covariance function of the signal $z(k)$, $K_z(k,s) = E[z(k)z(s)] = K_z(i)$, $i = k - s$, $0 \le i \le N$, the Yule-Walker equation for the AR parameters $\underline{a}_i$, $1 \le i \le M$, is given by

$$\begin{aligned} &K(k,k)\begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \vdots \\ \underline{a}_{M-1} \\ \underline{a}_M \end{bmatrix} = -\begin{bmatrix} K_z(1) \\ K_z(2) \\ \vdots \\ K_z(M-1) \\ K_z(M) \end{bmatrix}, \\ &K(k,k) \\ &= \begin{bmatrix} K_z(0) & K_z(1) & \cdots & K_z(M-2) & K_z(M-1) \\ K_z(1) & K_z(0) & \cdots & K_z(M-3) & K_z(M-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_z(M-2) & K_z(M-3) & \cdots & K_z(0) & K_z(1) \\ K_z(M-1) & K_z(M-2) & \cdots & K_z(1) & K_z(0) \end{bmatrix}. \end{aligned} \tag{5}$$

Here, we consider to develop the robust estimation technique for the signal $z(k)$ with the degraded measurement data $\breve{y}(k)$, which is generated by the state-space model (6) in actual environments.

$$\begin{aligned}&\breve{y}(k) = H(k)\breve{z}(k) + v(k), \breve{z}(k) = \breve{H}\breve{x}(k),\\ &\breve{x}(k+1) = \breve{\Phi}(k)\breve{x}(k) + \breve{\Gamma}\zeta(k),\\ &\breve{\Phi}(k) = \Phi + \Delta\Phi(k)\end{aligned} \quad (6)$$

In (6) the system matrix $\breve{\Phi}(k)$ contains the uncertain matrix $\Delta\Phi(k)$ additionally to the system matrix Φ, in comparison with the state-space model (1). Due to the uncertain quantity $\Delta\Phi(k)$, the trajectory of the state vector $\breve{x}(k)$ strays out of the nominal trajectory of $x(k)$. $\breve{z}(k)$ is the scalar degraded signal.

Let the sequence of the degraded signal $\breve{z}(k)$ be fitted to the AR model of the N-th order.

$$\begin{aligned}&\breve{z}(k) = -\breve{a}_1\breve{z}(k-1) - \breve{a}_2\breve{z}(k-2)\cdots\\ &-\breve{a}_N\breve{z}(k-N) + \breve{e}(k),\\ &E[\breve{e}(k)\breve{e}(s)] = \breve{Q}\delta_K(k-s)\end{aligned} \quad (7)$$

$\breve{z}(k)$ is expressed in terms of the state vector $\breve{x}(k)$ as

$$\begin{aligned}&\breve{z}(k) = \breve{H}\breve{x}(k), \breve{H} = [1 \quad 0 \quad 0 \quad \cdots \quad 0],\\ &\breve{x}(k) = \begin{bmatrix}\breve{x}_1(k)\\ \breve{x}_2(k)\\ \vdots\\ \breve{x}_{N-1}(k)\\ \breve{x}_N(k)\end{bmatrix} = \begin{bmatrix}\breve{z}(k)\\ \breve{z}(k+1)\\ \vdots\\ \breve{z}(k+N-2)\\ \breve{z}(k+N-1)\end{bmatrix}.\end{aligned} \quad (8)$$

Hence, the state equation for the state vector $\breve{x}(k)$ is described by

$$\begin{aligned}&\breve{x}(k+1) = \breve{\Phi}\breve{x}(k) + \breve{\Gamma}\zeta(k),\\ &E[\zeta(k)\zeta^T(s)] = \breve{Q}\delta_K(k-s),\\ &\breve{\Phi} = \begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -\breve{a}_N & -\breve{a}_{N-1} & -\breve{a}_{N-2} & \cdots & -\breve{a}_1\end{bmatrix},\\ &\Gamma = \begin{bmatrix}0\\0\\0\\ \vdots\\1\end{bmatrix},\\ &\zeta(k) = \breve{e}(k+N).\end{aligned} \quad (9)$$

The auto-covariance function $\breve{K}(k,s)$ of the state vector $\breve{x}(k)$ is assumed to have the semi-degenerate kernel form of

$$\begin{aligned}&\breve{K}(k,s) = \begin{cases}A(k)B^T(s), 0 \le s \le k,\\ B(k)A^T(s), 0 \le k \le s,\end{cases}\\ &A(k) = \breve{\Phi}^k, B^T(s) = \breve{\Phi}^{-s}\breve{K}(s,s).\end{aligned} \quad (10)$$

In terms of the auto-covariance function $K_{\breve{z}}(k,s) = E[\breve{z}(k)\breve{z}(s)]$ of the degraded signal $\breve{z}(k)$ in wide sense stationary stochastic systems, the auto-variance function $\breve{K}(k,k)$ of the state vector $\breve{x}(k)$ is expressed as follows.

$$\begin{aligned}&\breve{K}(k,k) = E\left[\begin{bmatrix}\breve{z}(k)\\ \breve{z}(k+1)\\ \vdots\\ \breve{z}(k+N-2)\\ \breve{z}(k+N-1)\end{bmatrix}\right.\\ &\times [\breve{z}(k) \quad \breve{z}(k+1) \quad \cdots\\ &\left.\breve{z}(k+N-2) \quad \breve{z}(k+N-1)]\right]\\ &= \begin{bmatrix}K_{\breve{z}}(0) & K_{\breve{z}}(-1) & \cdots & K_{\breve{z}}(-N+2) & K_{\breve{z}}(-N+1)\\ K_{\breve{z}}(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(-N+3) & K_{\breve{z}}(-N+2)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ K_{\breve{z}}(N-2) & K_{\breve{z}}(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(-1)\\ K_{\breve{z}}(N-1) & K_{\breve{z}}(N-2) & \cdots & K_{\breve{z}}(1) & K_{\breve{z}}(0)\end{bmatrix}\end{aligned} \quad (11)$$

Here, $K_{\breve{z}}(i) = K_{\breve{z}}(-i), 1 \le i \le N$. By using $K_{\breve{z}}(i)$, $0 \le i \le N$, the Yule-Walker equation for the AR parameters $\breve{a}_i$, $1 \le i \le N$, is formulated as

$$\begin{aligned}&\hat{K}(k,k)\begin{bmatrix}\breve{a}_1^T\\ \breve{a}_2^T\\ \vdots\\ \breve{a}_{N-1}^T\\ \breve{a}_N^T\end{bmatrix} = -\begin{bmatrix}K_z(1)\\ K_z(2)\\ \vdots\\ K_z(N-1)\\ K_z(N)\end{bmatrix},\\ &\hat{K}(k,k) = \begin{bmatrix}K_{\breve{z}}(0) & K_{\breve{z}}(-1) & \cdots & K_{\breve{z}}(-N+2) & K_{\breve{z}}(-N+1)\\ K_{\breve{z}}(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(-N+3) & K_{\breve{z}}(-N+2)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ K_{\breve{z}}(N-2) & K_{\breve{z}}(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(-1)\\ K_{\breve{z}}(N-1) & K_{\breve{z}}(N-2) & \cdots & K_{\breve{z}}(1) & K_{\breve{z}}(0)\end{bmatrix}.\end{aligned} \quad (12)$$

Let $K_{\underline{x}\breve{x}}(k,s) = E[\underline{x}(k)\breve{x}^T(s)]$ represent the cross-covariance function of the state vector $\underline{x}(k)$ with $\breve{x}(s)$. Let $K_{\underline{x}\breve{x}}(k,s)$ have the functional form of

$$K_{\underline{x}\breve{x}}(k,s)=\alpha(k)\beta^T(s), 0\le s\le k,\quad \alpha(k)=\underline{\Phi}^k, \beta^T(s)=\underline{\Phi}^{-s}K_{\underline{x}\breve{x}}(s,s) \tag{13}$$

with the system matrix $\underline{\Phi}$ for the state vector $\underline{x}(k)$.

Let the fixed-point smoothing estimate $\hat{\underline{x}}(k,L)$ of the state vector $\underline{x}(k)$ at the fixed point k be given by

$$\hat{\underline{x}}(k,L)=\sum_{i=1}^{L}h(k,i,L)\breve{y}(i) \tag{14}$$

as a sum of the products of the impulse response function $h(k,i,L)$ and the observed values $\breve{y}(i), 1\le i\le L$. Let us consider the least-squares estimation problem, which minimizes the mean-square value (MSV)

$$J=E[||\underline{x}(k)-\hat{\underline{x}}(k,L)||^2] \tag{15}$$

of the fixed-point smoothing error $\underline{x}(k)-\hat{\underline{x}}(k,L)$. From an orthogonal projection lemma [11]

$$\underline{x}(k)-\sum_{i=1}^{L}h(k,i,L)\breve{y}(i)\perp\breve{y}(s),\quad 1\le s\le L, \tag{16}$$

we obtain the Wiener-Hopf equation

$$E[\underline{x}(k)\breve{y}^T(s)]=\sum_{i=1}^{L}h(k,i,L)E[\breve{y}(i)\breve{y}(s)], \tag{17}$$

which the optimal impulse response function satisfies. In (16), '⊥' represents the notation of the orthogonality. From (6), (8) and (17), and taking into account of the relationship $E[\underline{x}(k)\breve{y}^T(s)]=K_{\underline{x}\breve{x}}(k,s)\breve{H}^T=K_{\underline{x}\breve{z}}(k,s)$,

$$h(k,s,L)R=K_{\underline{x}\breve{x}}(k,s)\breve{H}^TH^T(s)-\sum_{i=1}^{L}h(k,i,L)H(i)\breve{H}\breve{K}(i,s)\breve{H}^TH^T(s) \tag{18}$$

is obtained. Here, $K_{\underline{x}\breve{x}}(k,s)$ represents the cross-covariance function of the state vector $\underline{x}(k)$ with the degraded state $\breve{x}(s)$ as $E[\underline{x}(k)\breve{x}^T(s)]$.

3 Robust RLS Wiener fixed-point smoothing and filtering algorithms

Based on the assumptions, in section 2, on the robust estimation problem for the observation equation with the linear amplitude modulation for the signal $z(k)$, Theorem 1 presents the robust RLS Wiener fixed-point smoothing and filtering algorithms.

Theorem 1 Let the observation equation, concerned with the linear amplitude modulation for the signal $z(k)$, be given by (1). Then the robust RLS Wiener fixed-point smoothing and filtering algorithms consist of (19)-(29) in linear discrete-time stochastic systems with the wide-sense stationarity. Here, the following information is used. The observation vector $\underline{H}$ in (3) and the system matrix $\underline{\Phi}$ in (4). The linear modulating function $H(k)$. The observation matrix $\breve{H}$ in (8) and the system matrix $\breve{\Phi}$ in (9). The cross-variance function $K_{\underline{x}\breve{x}}(k,k)$ of $\underline{x}(k)$ with $\breve{x}(k)$. The variance $\breve{K}(k,k)$ of $\breve{x}(k)$. The variance R of the observation noise. The degraded observed value $\breve{y}(k)$.

Fixed-point smoothing estimate of the signal $z(k)$ at the fixed point k: $\hat{z}(k,L)=\underline{H}\hat{\underline{x}}(k,L)$

Fixed-point smoothing estimate of the state vector $\underline{x}(k)$ at the fixed point k: $\hat{\underline{x}}(k,L)$

$$\hat{\underline{x}}(k,L)=\hat{\underline{x}}(k,L-1)+h(k,L,L)(\breve{y}(L)-H(L)\breve{H}\breve{\Phi}\hat{\breve{x}}(L-1,L-1)) \tag{19}$$

Smoother gain: $h(k,L,L)$

$$h(k,L,L)=(K_{\underline{x}\breve{x}}(k,k)(\breve{\Phi}^T)^{L-k}\breve{H}^TH^T(L)-q(k,L-1)\breve{\Phi}^T\breve{H}^TH^T(L))\times(R+H(L)\breve{H}(\breve{K}(L,L)-\breve{\Phi}S_0(L-1)\breve{\Phi}^T)\breve{H}^TH(L))^{-1} \tag{20}$$

$$q(k,L)=q(k,L-1)\breve{\Phi}^T+h(k,L,L)H(L)\breve{H}(\breve{K}(L,L)-\breve{\Phi}S_0(L-1)\breve{\Phi}^T),\quad q(k,k)=S(k) \tag{21}$$

Filtering estimate of the signal $z(k)$: $\hat{z}(k,k)=\underline{H}\hat{\underline{x}}(k,k)$

Filtering estimate of the state vector $\underline{x}(k)$: $\hat{\underline{x}}(k,k)$

$$\begin{aligned}\hat{\underline{x}}(k,k) &= \underline{\Phi}\hat{\underline{x}}(k-1,k-1) + G(k)(\breve{y}(k)\\ &-H(k)\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)), \quad \hat{\underline{x}}(0,0)=0\end{aligned} \tag{22}$$

Filtering estimate of the state vector $\breve{x}(k)$: $\hat{\breve{x}}(k,k)$

$$\begin{aligned}\hat{\breve{x}}(k,k) &= \breve{\Phi}\hat{\breve{x}}(k-1,k-1) + g(k)(\breve{y}(k)\\ &-H(k)\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)), \quad \hat{\breve{x}}(0,0)=0\end{aligned} \tag{23}$$

One-step ahead prediction estimate of the signal $z(k)$: $\hat{z}(k,k-1)$

$$\hat{z}(k,k-1) = \underline{H\hat{x}}(k,k-1) \tag{24}$$

One-step ahead prediction estimate of the state vector $\underline{x}(k)$: $\hat{\underline{x}}(k,k-1)$

$$\hat{\underline{x}}(k,k-1) = \underline{\Phi\hat{x}}(k-1,k-1) \tag{25}$$

Cross-variance function of $\hat{\underline{x}}(k,k)$ with $\hat{\breve{x}}(k,k)$: $S(k)$

$$\begin{aligned}S(k) &= \underline{\Phi}S(k-1)\breve{\Phi}^T\\ &+G(k)H(k)\breve{H}(\breve{K}(k,k)\\ &-\breve{\Phi}S_0(k-1)\breve{\Phi}^T), \quad S(0)=0\end{aligned} \tag{26}$$

Auto-variance function of $\hat{\breve{x}}(k,k)$: $S_0(k)$

$$\begin{aligned}S_0(k) &= \breve{\Phi}S_0(k-1)\breve{\Phi}^T\\ &+g(k)H(k)\breve{H}(\breve{K}(k,k)\\ &-\breve{\Phi}S_0(k-1)\breve{\Phi}^T), \quad S_0(0)=0\end{aligned} \tag{27}$$

Filter gain for $\hat{\underline{x}}(k,k)$: $G(k)$

$$\begin{aligned}G(k) &= (K_{\underline{x}\breve{x}}(k,k)\breve{H}^T H^T(k)\\ &-\underline{\Phi}S(k-1)\breve{\Phi}^T\breve{H}^T H^T(k))\\ &\times(R+H(k)\breve{H}(\breve{K}(k,k)\\ &-\breve{\Phi}S_0(k-1)\breve{\Phi}^T)\breve{H}^T H^T(k))^{-1}\end{aligned} \tag{28}$$

Filter gain for $\hat{\breve{x}}(k,k)$: $g(k)$

$$\begin{aligned}g(k) &= (\breve{K}(k,k)\breve{H}^T H^T(k)\\ &-\breve{\Phi}S_0(k-1)\breve{\Phi}^T\breve{H}^T H^T(k))\\ &\times(R+H(k)\breve{H}(\breve{K}(k,k)\\ &-\breve{\Phi}S_0(k-1)\breve{\Phi}^T)\breve{H}^T H^T(k))^{-1}\end{aligned} \tag{29}$$

Proof of Theorem 1 is deferred to the appendix.

4 Robust extended recursive Wiener estimation algorithms for observation mechanism with nonlinear modulation

Let an observation equation with the nonlinear modulation of the signal $z(k)$ be given by

$$\begin{aligned}&y(k) = f(z(k),k) + v(k),\\ &z(k) = Cx(k),\end{aligned} \tag{30}$$

where the signal $z(k)$ and the observation noise $v(k)$ have the same stochastic properties as those in section 2.

Likewise the design method of the extended Kalman filter, in the design of the robust extended recursive Wiener estimators, the modulating function is put as $H(k) = \frac{\partial f(z(k),k)}{\partial z(k)}\Big|_{z(k)=\hat{z}(k,k-1)}$ in Theorem 1 after expanding the nonlinear observation function in a first-order Taylor series about $\hat{z}(k,k-1)$ [1]. Here, $\hat{z}(k,k-1) = \underline{H\Phi\hat{x}}(k-1,k-1)$ represents the one-step ahead prediction estimate of the signal $z(k)$. Also, $H(L)\breve{H}\breve{\Phi}\hat{\breve{x}}(L-1,L-1)$ and $H(k)\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)$ in Theorem 1 are replaced with $f(\breve{H}\breve{\Phi}\hat{\breve{x}}(L-1,L-1),L)$ and $f(\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1),k)$ respectively.

Consequently, the robust extended recursive Wiener fixed-point smoothing and filtering algorithms in the case of the observation equation, with the nonlinear modulation of the signal $z(k)$, is summarized in Theorem 2. It is noted that the proposed robust extended recursive Wiener estimators are sub-optimal because of the Taylor series approximation of the nonlinear modulating function $f(z(k),k)$ of the signal $z(k)$.

Theorem 2 Let the observation equation, with the nonlinear modulating function $f(z(k),k)$ of the signal $z(k)$, be given by (30). Then the robust extended recursive Wiener fixed-point smoothing and filtering algorithms consist of (31)-(42) in discrete-time wide-sense stationary stochastic systems. Here, the following information is used. The observation vector $\underline{H}$ in (3) and the system matrix $\underline{\Phi}$ in (4). The nonlinear modulating function $f(z(k),k)$ and the function $H(k)$ given by (42). The observation vector $\breve{H}$ in (8) and the system matrix $\breve{\Phi}$ in (9). The cross-variance function $K_{\underline{x}\breve{x}}(k,k)$ of $\underline{x}(k)$ with $\breve{x}(k)$. The variance $\breve{K}(k,k)$ of $\breve{x}(k)$. The variance R of the observation noise. The degraded observed value $\breve{y}(k)$.

Fixed-point smoothing estimate of the signal $z(k)$ at the fixed point k: $\hat{z}(k,L)=\underline{H}\underline{\hat{x}}(k,L)$

Fixed-point smoothing estimate of the state vector $\underline{x}(k)$ at the fixed point k: $\underline{\hat{x}}(k,L)$

$$\begin{aligned}&\underline{\hat{x}}(k,L)=\underline{\hat{x}}(k,L-1)+h(k,L,L)(\breve{y}(L)\\&-f(\breve{H}\breve{\Phi}\hat{\breve{x}}(L-1,L-1),L))\end{aligned}\tag{31}$$

Smoother gain: $h(k,L,L)$

$$\begin{aligned}&h(k,L,L)=(K_{\underline{x}\breve{x}}(k,k)(\breve{\Phi}^T)^{L-k}\breve{H}^T H^T(L)\\&-q(k,L-1)\breve{\Phi}^T\breve{H}^T H^T(L))\\&\times(R+H(L)\breve{H}(\breve{K}(L,L)\\&-\breve{\Phi}S_0(L-1)\breve{\Phi}^T)\breve{H}^T H^T(L))^{-1}\end{aligned}\tag{32}$$

$$\begin{aligned}&q(k,L)=q(k,L-1)\breve{\Phi}^T\\&+h(k,L,L)H(L)\breve{H}(\breve{K}(L,L)\\&-\breve{\Phi}S_0(L-1)\breve{\Phi}^T),\\&q(k,k)=S(k)\end{aligned}\tag{33}$$

Filtering estimate of the signal $z(k)$: $\hat{z}(k,k)=\underline{H}\underline{\hat{x}}(k,k)$

Filtering estimate of the state vector $\underline{x}(k)$: $\underline{\hat{x}}(k,k)$

$$\begin{aligned}&\underline{\hat{x}}(k,k)=\underline{\Phi}\underline{\hat{x}}(k-1,k-1)+G(k)(\breve{y}(k)\\&-f(\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1),k)),\quad \underline{\hat{x}}(0,0)=0\end{aligned}\tag{34}$$

Filtering estimate of the state vector $\breve{x}(k)$: $\hat{\breve{x}}(k,k)$

$$\begin{aligned}&\hat{\breve{x}}(k,k)=\breve{\Phi}\hat{\breve{x}}(k-1,k-1)+g(k)(\breve{y}(k)\\&-f(\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1),k)),\quad \hat{\breve{x}}(0,0)=0\end{aligned}\tag{35}$$

One-step ahead prediction estimate of the signal $z(k)$: $\hat{z}(k,k-1)$

$$\hat{z}(k,k-1)=\underline{H}\underline{\hat{x}}(k,k-1)\tag{36}$$

One-step ahead prediction estimate of the state vector $\underline{x}(k)$: $\underline{\hat{x}}(k,k-1)$

$$\underline{\hat{x}}(k,k-1)=\underline{\Phi}\underline{\hat{x}}(k-1,k-1)\tag{37}$$

Cross-variance function of $\underline{\hat{x}}(k,k)$ with $\hat{\breve{x}}(k,k)$: $S(k)$

$$\begin{aligned}&S(k)=\underline{\Phi}S(k-1)\breve{\Phi}^T\\&+G(k)H(k)\breve{H}(\breve{K}(k,k)-\breve{\Phi}S_0(k-1)\breve{\Phi}^T),\\&S(0)=0\end{aligned}\tag{38}$$

Auto-variance function of $\hat{\breve{x}}(k,k)$: $S_0(k)$

$$\begin{aligned}&S_0(k)=\breve{\Phi}S_0(k-1)\breve{\Phi}^T\\&+g(k)H(k)\breve{H}(\breve{K}(k,k)-\breve{\Phi}S_0(k-1)\breve{\Phi}^T),\\&S_0(0)=0\end{aligned}\tag{39}$$

Filter gain for $\underline{\hat{x}}(k,k)$: $G(k)$

$$\begin{aligned}&G(k)=(K_{\underline{x}\breve{x}}(k,k)\breve{H}^T H(k)\\&-\underline{\Phi}S(k-1)\breve{\Phi}^T\breve{H}^T H^T(k))\\&\times(R+H(k)\breve{H}(\breve{K}(k,k)\\&-\breve{\Phi}S_0(k-1)\breve{\Phi}^T)\breve{H}^T H^T(k))^{-1}\end{aligned}\tag{40}$$

Filter gain for $\hat{\breve{x}}(k,k)$: $g(k)$

$$\begin{aligned}&g(k)=(\breve{K}(k,k)\breve{H}^T H(k)\\&-\breve{\Phi}S_0(k-1)\breve{\Phi}^T\breve{H}^T H^T(k))\\&\times(R+H(k)\breve{H}(\breve{K}(k,k)\\&-\breve{\Phi}S_0(k-1)\breve{\Phi}^T)\breve{H}^T H^T(k))^{-1}\end{aligned}\tag{41}$$

Here, the function $H(k)$ is given by

$$\begin{aligned}&H(k)=\left.\frac{\partial f(z(k),k)}{\partial z(k)}\right|_{z(k)=\hat{z}(k,k-1)},\\&\hat{z}(k,k-1)=\underline{H}\underline{\Phi}\underline{\hat{x}}(k-1,k-1).\end{aligned}\tag{42}$$

A necessary condition for the stability of the robust extended recursive Wiener estimators is given by $R+H(k)\breve{H}(\breve{K}(k,k)-\breve{\Phi}S_0(k-1)\breve{\Phi}^T)\breve{H}^T H^T(k)>0$.

5 A numerical simulation example

Let a scalar observation equation with the nonlinear mechanism of the signal $z(k)$ be given by

$$\begin{aligned} &y(k) = f(z(k), k) + v(k), z(k) = Cx(k), \\ &f(z(k), k) = \cos(2\pi f_c k\Delta + m_A z(k)), \\ &f_c = 1{,}000(Hz), \Delta = 0.000090703, \\ &m_A = 1.2. \end{aligned} \quad (43)$$

The nonlinear function in (43) appears in the phase modulation of analogue communication systems [1]. Here, f_c, Δ and m_A represent the carrier frequency, the sampling period of the signal $z(k)$ and the phase sensitivity respectively. The function $H(k)$ in (42) becomes

$$\begin{aligned} H(k) &= \left.\frac{\partial f(z(k), k)}{\partial z(k)}\right|_{z(k)=\hat{z}(k|k-1)} \\ &= -m_A \sin(2\pi f_c k\Delta + m_A \hat{z}(k|k-1)). \end{aligned} \quad (44)$$

Let $v(k)$ be the white Gaussian observation noise with the mean zero and the variance R, i. e. $N(0, R)$. Let the signal process be expressed in terms of the AR model of the order 2.

$$\begin{aligned} &z(k) = -\underline{a}_1 z(k-1) - \underline{a}_2 z(k-2) \\ &+\underline{e}(k), E[\underline{e}(k)\underline{e}(s)] = \underline{Q}\delta_K(k-s), \\ &\underline{Q} = 0.5^2. \end{aligned} \quad (45)$$

Let $z(k)$ be expressed in terms of the state vector $\underline{x}(k)$ as follows.

$$\begin{aligned} &z(k) = \underline{H}\underline{x}(k), \underline{H} = [1 \quad 0], \\ &\underline{x}(k) = \begin{bmatrix} \underline{x}_1(k) \\ \underline{x}_2(k) \end{bmatrix} = \begin{bmatrix} z(k) \\ z(k+1) \end{bmatrix} \end{aligned} \quad (46)$$

Then the state equation, corresponding to the AR model (45), is described by

$$\begin{aligned} &\underline{x}(k+1) = \underline{\Phi}\underline{x}(k) + \underline{\Gamma}\underline{w}(k), \\ &E[\underline{w}(k)\underline{w}^T(s)] = \underline{Q}\delta_K(k-s), \\ &\underline{\Phi} = \begin{bmatrix} 0 & 1 \\ -\underline{a}_2 & -\underline{a}_1 \end{bmatrix}, \underline{\Gamma} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ &\underline{w}(k) = \underline{e}(k+2), \underline{a}_1 = -0.1, \underline{a}_2 = -0.8. \end{aligned} \quad (47)$$

Let the degraded measurement data $\breve{y}(k)$ be generated by the state-space model.

$$\begin{aligned} &\breve{y}(k) = H(k)\breve{z}(k) + v(k), \breve{z}(k) = \breve{H}\breve{x}(k), \\ &\breve{x}(k+1) = \breve{\Phi}\breve{x}(k) + \breve{\Gamma}\zeta(k), \\ &\breve{\Phi}(k) = \underline{\Phi} + \Delta\Phi(k), \\ &\Delta\Phi(k) = \begin{bmatrix} 0 & 0 \\ 0.08 & 0.05 \end{bmatrix}. \end{aligned} \quad (48)$$

In (48) the system matrix $\breve{\Phi}(k)$ contains the uncertain matrix $\Delta\Phi(k)$ additionally to the system matrix $\underline{\Phi}$, in comparison with the state-space model (47). $\breve{z}(k)$ is the degraded signal. Due to the uncertain quantity $\Delta\Phi(k)$, the trajectory of the state vector $\breve{x}(k)$ strays out of the nominal trajectory of $\underline{x}(k)$.

Let the sequence of the degraded signal $\breve{z}(k)$ be fitted to the 5-th order AR model.

$$\begin{aligned} &\breve{z}(k) = -\breve{a}_1\breve{z}(k-1) - \breve{a}_2\breve{z}(k-2)\cdots \\ &-\breve{a}_N\breve{z}(k-N) + \breve{e}(k), \\ &E[\breve{e}(k)\breve{e}(s)] = \breve{Q}\delta_K(k-s), N = 5 \end{aligned} \quad (49)$$

$\breve{z}(k)$ is expressed in terms of the state vector $\breve{x}(k)$ as

$$\begin{aligned} &\breve{z}(k) = \breve{H}\breve{x}(k), \breve{H} = [1 \quad 0 \quad 0 \quad 0 \quad 0], \\ &\breve{x}(k) = \begin{bmatrix} \breve{x}_1(k) \\ \breve{x}_2(k) \\ \breve{x}_3(k) \\ \breve{x}_4(k) \\ \breve{x}_5(k) \end{bmatrix} = \begin{bmatrix} \breve{z}(k) \\ \breve{z}(k+1) \\ \breve{z}(k+2) \\ \breve{z}(k+3) \\ \breve{z}(k+4) \end{bmatrix}. \end{aligned} \quad (50)$$

Hence, the state equation for the state vector $\breve{x}(k)$ is described by

$$\begin{aligned} &\breve{x}(k+1) = \breve{\Phi}\breve{x}(k) + \breve{\Gamma}\zeta(k), \\ &E[\zeta(k)\zeta^T(s)] = \breve{Q}\delta_K(k-s), \\ &\breve{\Phi} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\breve{a}_5 & -\breve{a}_4 & -\breve{a}_3 & -\breve{a}_2 & -\breve{a}_1 \end{bmatrix}, \\ &\breve{\Gamma} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \\ &\zeta(k) = \breve{e}(k+5). \end{aligned} \quad (51)$$

Substituting $\underline{H}$, $\breve{H}$, $\underline{\Phi}$, $\breve{\Phi}$, $H(k)$, $\breve{y}(k)$, $K_{\underline{x}\breve{x}}(k,k)$, $\breve{K}(k,k)$ and $f(\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1),k)$ into the robust extended recursive Wiener filtering and fixed-point smoothing algorithms of Theorem 2, the filtering and fixed-point smoothing estimates of the signal $z(k)$ are calculated recursively.

Fig.1 illustrates the signal $z(k)$, the filtering estimate $\hat{z}(k,k)$ and the fixed-point smoothing estimate $\hat{z}(k,k+Lag), Lag=5$, vs. k for the white Gaussian observation noise $N(0,0.3^2)$. Fig.2 compares the mean-square values (MSVs) of the estimation errors by the robust extended recursive Wiener filter and fixed-point smoother with those by the extended Wiener recursive filter and fixed-point smoother [1] vs. Lag, $0 \le Lag \le 10$, for the white Gaussian observation noises $N(0,0.3^2)$, $N(0,0.5^2)$ and $N(0,0.7^2)$. From Fig.2, it is seen that the estimation accuracy of the robust extended recursive Wiener filter and fixed-point smoother is superior to the extended recursive Wiener estimators [1] for the respective observation noise. Here, the MSVs of the estimation errors are calculated by $\sum_{k=1}^{600}(z(k)-\hat{z}(k,k+Lag))^2/600$, $1 \le Lag \le 10$, for the fixed-point smoothing errors and $\sum_{k=1}^{600}(z(k)-\hat{z}(k,k))^2/600$ for the filtering errors.

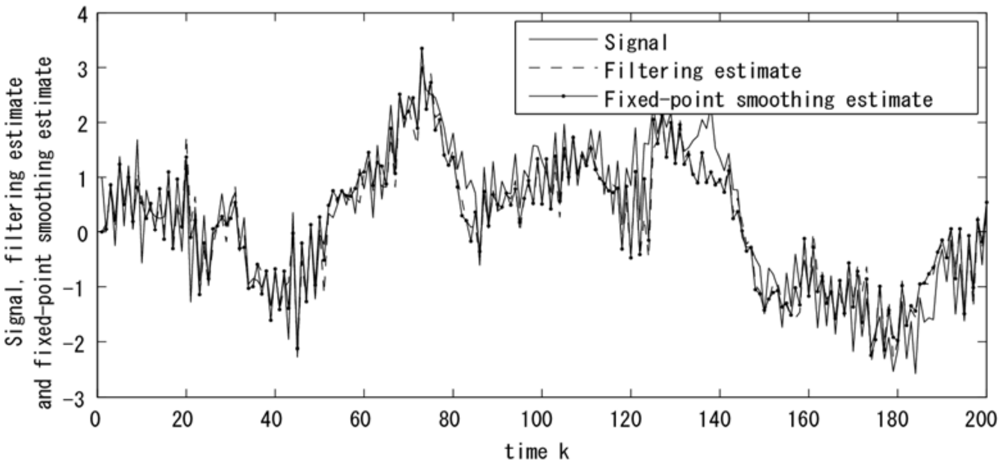

Fig.1 Signal $z(k)$, filtering estimate $\hat{z}(k,k)$ and fixed-point smoothing estimate $\hat{z}(k,k+5)$ for white Gaussian observation noise $N(0,0.3^2)$ vs. k.

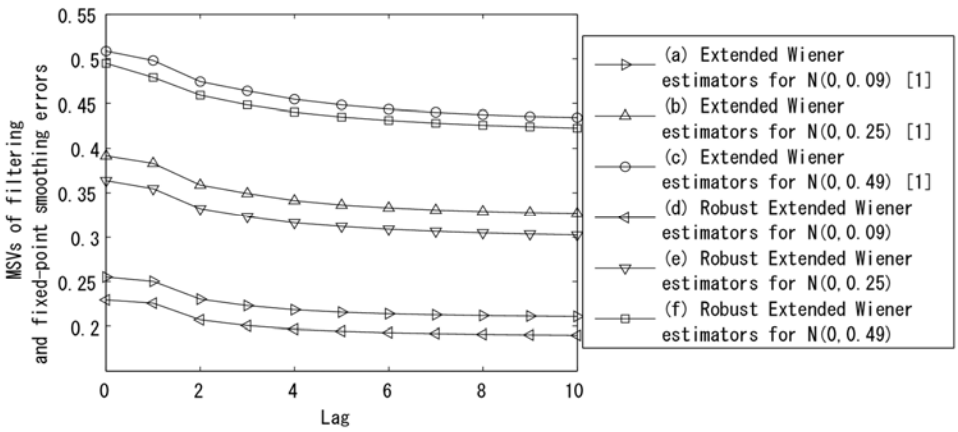

Fig.2. Comparison of MSVs of estimation errors by robust extended recursive Wiener filter and fixed-point smoother with those by extended recursive Wiener filter and fixed-point smoother [1] vs. Lag, $0 \le Lag \le 10$ for white Gaussian observation noises $N(0,0.3^2)$, $N(0,0.5^2)$ and $N(0,0.7^2)$.

6 Conclusion

This paper, as an extension of the linear robust RLS Wiener filter and fixed-point smoother in linear discrete-time stochastic systems, originally proposed the robust extended recursive Wiener filter and fixed-point smoother for estimating the signal. It is a characteristic in this paper that the signal is modulated with the nonlinear mechanism. The observation noise is additional white noise. The system matrix in the state equation contains uncertain parameters.

In the simulation example, it is shown that the proposed robust extended recursive Wiener filter and fixed-point smoother are superior in estimation accuracy to the extended recursive Wiener estimators.

Appendix: Proof of Theorem 1

From (18) the optimal impulse response function satisfies

$$h(k,s,L)R = K_{\underline{x}\breve{x}}(k,s)\breve{H}^T H^T(s) - \sum_{i=1}^{L} h(k,i,L)H(i)\breve{H}\breve{K}(i,s)\breve{H}^T H^T(s). \quad \text{(A-1)}$$

Subtracting $h(k,s,L-1)R$ from $h(k,s,L)R$, we have

$$(h(k,s,L) - h(k,s,L-1)R = -h(k,L,L)H(L)\breve{H}\breve{K}(L,s)\breve{H}^T H^T(s) - \sum_{i=1}^{L-1}(h(k,i,L) - h(k,i,L-1)) \times H(i)\breve{H}\breve{K}(i,s)\breve{H}^T H^T(s).$$

By introducing

$$J_0(s,L)R = \breve{\Phi}^{-s}\breve{K}(s,s)\breve{H}^T H^T(s) - \sum_{i=1}^{L} J_0(i,L)H(i)\breve{H}\breve{K}(i,s)\breve{H}^T H^T(s), \quad \text{(A-2)}$$

$$h(k,s,L) - h(k,s,L-1) = -h(k,L,L)H(L)\breve{H}\breve{\Phi}^L J_0(s,L-1) \quad \text{(A-3)}$$

is obtained. Subtracting $J_0(s,L-1)R$ from $J_0(s,L)R$, we have

$$(J_0(s,L) - J_0(s,L-1))R = -J_0(L,L)H(L)\breve{H}\breve{K}(L,s)\breve{H}^T H^T(s) - \sum_{i=1}^{L-1}(J_0(i,L) - J_0(i,L-1)) \times H(i)\breve{H}\breve{K}(i,s)\breve{H}^T H^T(s). \quad \text{(A-4)}$$

From (A-2) and (A-4), we have

$$J_0(s,L) - J_0(s,L-1) = -J_0(L,L)H(L)\breve{H}\breve{\Phi}^L J_0(s,L-1). \quad \text{(A-5)}$$

From (14) the filtering estimate of $\underline{x}(k)$ is given by

$$\hat{\underline{x}}(k,k) = \sum_{i=1}^{k} h(k,i,k)\breve{y}(i). \quad \text{(A-6)}$$

The optimal impulse response function $h(k,s,k)$ in the filtering problem satisfies

$$h(k,s,k)R = K_{\underline{x}\breve{x}}(k,k)\breve{H}^T H^T(k) - \sum_{i=1}^{k} h(k,i,k)H(i)\breve{H}\breve{K}(i,k)\breve{H}^T H^T(k). \quad \text{(A-7)}$$

By introducing

$$J(s,k)R = \underline{\Phi}^{-s}K_{\underline{x}\breve{x}}(s,s)\breve{H}^T H^T(s) - \sum_{i=1}^{k} J(i,k)H(i)\breve{H}\breve{K}(i,s)\breve{H}^T H^T(s), \quad \text{(A-8)}$$

$$h(k,s,k) = \alpha(k)J(s,k), \alpha(k) = \underline{\Phi}^k \quad \text{(A-9)}$$

is obtained. Subtracting $J(s,k-1)R$ from $J(s,k)R$, we have

$$(J(s,k) - J(s,k-1))R = -J(k,k)H(k)\breve{H}\breve{K}(k,s)\breve{H}^T H^T(s) - \sum_{i=1}^{k}(J(i,k) - J(i,j-1)) \times H(i)\breve{H}\breve{K}(i,s)\breve{H}^T H^T(s). \quad \text{(A-10)}$$

From (A-2) and (A-10), it follows that

$$J(s,k)-J(s,k-1) = -J(k,k)H(k)\breve{H}\breve{\Phi}^k J_0(s,k-1). \quad \text{(A-11)}$$

From (A-8) $J(k,k)$ is expressed as follows.

$$J(k,k)R = \underline{\Phi}^{-k} K_{\underline{x}\breve{x}}(k,k)\breve{H}^T H^T(k) - \sum_{i=1}^{k} J(i,k)H(i)\breve{H}\breve{K}(i,k)\breve{H}^T H^T(k) = \underline{\Phi}^{-k} K_{\underline{x}\breve{x}}(k,k)\breve{H}^T H^T(k) - r(k)A^T(k)\breve{H}^T H^T(k). \quad \text{(A-12)}$$

Here, $r(k)$ is given by

$$r(k) = \sum_{i=1}^{k} J(i,k)H(i)\breve{H}B(i). \quad \text{(A-13)}$$

Subtracting $r(k-1)$ from $r(k)$ and using (A-11), it follows that

$$r(k)-r(k-1) = J(k,k)H(k)\breve{H}B(k) - J(k,k)H(k)\breve{H}\breve{\Phi}^k \times \sum_{i=1}^{k-1} J_0(i,k-1)H(i)\breve{H}B(i) = J(k,k)H(k)\breve{H}(B(k) - \breve{\Phi}^k r_0(k-1)), \quad r(0)=0. \quad (A-14)$$

Here, $r_0(k)$ is given by

$$r_0(k) = \sum_{i=1}^{k} J_0(i,k)H(i)\breve{H}B(i). \quad \text{(A-15)}$$

From (A-12) and (A-14), we have

$$J(k,k) = (\underline{\Phi}^{-k} K_{\underline{x}\breve{x}}(k,k)\breve{H}^T H^T(k) - r(k-1)A^T(k)\breve{H}^T H^T(k)) \times (R + H(k)\breve{H}(\breve{K}(k,k) - \breve{\Phi}^k r_0(k-1)(\breve{\Phi}^T)^k)\breve{H}^T H^T(k))^{-1} = (\underline{\Phi}^{-k} K_{\underline{x}\breve{x}}(k,k)\breve{H}^T H^T(k) - r(k-1)A^T(k)\breve{H}^T H^T(k)) \times (R + H(k)\breve{H}(\breve{K}(k,k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T)\breve{H}^T H^T(k))^{-1}. \quad \text{(A-16)}$$

after some manipulations. Here,

$$S_0(k) = A(k)r_0(k)A^T(k), A(k) = \breve{\Phi}^k. \quad \text{(A-17)}$$

Subtracting $r_0(k-1)$ from $r_0(k)$ and using (A-5), we have

$$r_0(k)-r_0(k-1) = J_0(k,k)H(k)\breve{H}B(k) + \sum_{i=1}^{k-1}(J_0(i,k) - J_0(i,k-1))H(i)\breve{H}B(i) = J_0(k,k)H(k)\breve{H}(B(k) - \breve{\Phi}^k r_0(k-1)), \quad r_0(0)=0. \quad \text{(A-18)}$$

From (A-2) and (A-15), $J_0(k,k)$ satisfies

$$J_0(k,k)R = \breve{\Phi}^{-k}\breve{K}(k,k)\breve{H}^T H^T(k) - \sum_{i=1}^{k} J_0(i,k)H(i)\breve{H}\breve{K}(i,k)\breve{H}^T H^T(k) = \breve{\Phi}^{-k}\breve{K}(k,k)\breve{H}^T H^T(k) - r_0(k)A^T(k)\breve{H}^T H^T(k). \quad (A-19)$$

From (A-18) and (A-19), after some manipulations, we have

$$J_0(k,k) = (\breve{\Phi}^{-k}\breve{K}(k,k)\breve{H}^T H^T(k) - r_0(k-1)A^T(k)\breve{H}^T H^T(k)) \times (R + H(k)\breve{H}(\breve{K}(k,k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T)\breve{H}^T H^T(k))^{-1}. \quad (A-20)$$

Substituting (A-18) into (A-17), we have

$$S_0(k) = A(k)r_0(k)A^T(k) = \breve{\Phi}^k(r_0(k-1) + J_0(k,k)H(k)\breve{H}(B(k) - \breve{\Phi}^k r_0(k-1))(\breve{\Phi}^T)^k = \breve{\Phi}S_0(k-1)\breve{\Phi}^T + g(k)H(k)\breve{H}(\breve{K}(k,k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T), \quad S_0(0)=0. \quad (A-21)$$

Here, $g(k)$ is given by

$$g(k) = \breve{\Phi}^k J_0(k,k). \quad \text{(A-22)}$$

Substituting (A-20) into (A-22), $g(k)$ is calculated by

$$g(k) = (\breve{K}(k,k)\breve{H}^T H^T(k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T\breve{H}^T H^T(k)) \times (R + H(k)\breve{H}(\breve{K}(k,k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T)\breve{H}^T H^T(k))^{-1}. \quad \text{(A-23)}$$

From (A-6) and (A-9), the filtering estimate is calculated by

$$\underline{\hat{x}}(k,k) = \alpha(k)\sum_{i=1}^{k} J(i,k)\breve{y}(i) = \alpha(k)e(k) = \underline{\Phi}^k e(k). \quad \text{(A-24)}$$

Here, $e(k)$ is given by

$$e(k) = \sum_{i=1}^{k} J(i,k)\breve{y}(i). \quad \text{(A-25)}$$

Subtracting $e(k-1)$ from $e(k)$ and using (A-11), it follows that

$$e(k) - e(k-1) = J(k,k)\breve{y}(k) + \sum_{i=1}^{k-1}(J(i,k) - J(i,k-1))\breve{y}(i) = J(k,k)\breve{y}(k) - J(k,k)H(k)\breve{H}\breve{\Phi}^k \times \sum_{i=1}^{k-1} J_0(i,k-1)\breve{y}(i) = J(k,k)(\breve{y}(k) - H(k)\breve{H}\breve{\Phi}^k e_0(k-1)), e(0) = 0. \quad \text{(A-26)}$$

Here, $e_o(k)$ is given by

$$e_0(k) = \sum_{i=1}^{k} J_0(i,k)\breve{y}(i). \quad (A-27)$$

Subtracting $e_0(k-1)$ from $e_0(k)$ and using (A-5), it follows that

$$e_0(k) - e_0(k-1) = J_0(k,k)\breve{y}(k) + \sum_{i=1}^{k-1}(J_0(i,k) - J_0(i,k-1))\breve{y}(i) = J_0(k,k)(\breve{y}(k) - H(k)\breve{H}\breve{\Phi}^k \sum_{i=1}^{k-1} J_0(i,k-1)\breve{y}(i)) = J_0(k,k)(\breve{y}(k) - H(k)\breve{H}\breve{\Phi}^k e_0(k-1)), e_0(0) = 0. \quad \text{(A-28)}$$

From (A-24) and (A-26), the filtering estimate $\hat{x}(k,k)$ is developed as follows.

$$\underline{\hat{x}}(k,k) = \underline{\Phi}^k(e(k-1) + J(k,k)(\breve{y}(k) - H(k)\breve{H}\breve{\Phi}^k e_0(k-1)) = \underline{\Phi}\underline{\hat{x}}(k-1,k-1) + G(k)(\breve{y}(k) - H(k)\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)), \underline{\hat{x}}(0,0) = 0. \quad \text{(A-29)}$$

Here, the filtering estimate $\hat{\breve{x}}(k,k)$ of $\breve{x}(k)$ is given by

$$\hat{\breve{x}}(k,k) = \breve{\Phi}^k e_0(k). \quad \text{(A-30)}$$

In (A-29) the filter gain $G(k)$ is given by

$$G(k) = \underline{\Phi}^k J(k,k). \quad \text{(A-31)}$$

Substituting (A-16) into (A-31), it follows that

$$G(k) = (K_{\underline{x}\breve{x}}(k,k)\breve{H}^T H^T(k) - \Phi^k r(k-1)A^T(k)\breve{H}^T H^T(k)) \times (R + H(k)\breve{H}(\breve{K}(k,k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T)\breve{H}^T H^T(k))^{-1} = (K_{\underline{x}\breve{x}}(k,k)\breve{H}^T H^T(k) - \Phi S(k-1)\breve{\Phi}^T\breve{H}^T H^T(k)) \times (R + H(k)\breve{H}(\breve{K}(k,k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T)\breve{H}^T H^T(k))^{-1}. \quad \text{(A-32)}$$

Here, $S(k)$ is given by

$$S(k) = \Phi^k r(k)(\breve{\Phi}^T)^k. \quad \text{(A-33)}$$

Substituting (A-28) into (A-30), it follows that

$$\begin{aligned}&\hat{\breve{x}}(k,k)=\breve{\Phi}^k e_0(k-1)\\&+\breve{\Phi}^k J_0(k,k)(\breve{y}(k)\\&-H(k)\breve{H}\breve{\Phi}^k e_0(k-1))\\&=\breve{\Phi}\hat{\breve{x}}(k-1,k-1)\\&+g(k)(\breve{y}(k)\\&-H(k)\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)),\\&\hat{\breve{x}}(0,0)=0.\end{aligned} \tag{A-34}$$

Substituting (A-14) into (A-33) and using (A-31), it follows that

$$\begin{aligned}&S(k)=\underline{\Phi}^k(r(k-1)\\&+J(k,k)H(k)\breve{H}(B(k)\\&-\breve{\Phi}^k r_0(k-1))(\breve{\Phi}^T)^k\\&=\underline{\Phi}S(k-1)\breve{\Phi}^T\\&+G(k)H(k)\breve{H}(\breve{K}(k,k)\\&-\breve{\Phi}S(k-1)\breve{\Phi}^T),\\&S(0)=0.\end{aligned} \tag{A-35}$$

From (A-1) $h(k,L,L)$ satisfies

$$\begin{aligned}&h(k,L,L)R=K_{\underline{x}\breve{x}}(k,L)\breve{H}^T H^T(L)\\&-\sum_{i=1}^{L}h(k,i,L)H(i)\breve{H}\breve{K}(i,L)\breve{H}^T H^T(L)\\&=K_{\underline{x}\breve{x}}(k,k)(\breve{\Phi}^T)^{L-k}\breve{H}^T H^T(L)\\&-P(k,L)(\breve{\Phi}^T)^L\breve{H}^T H^T(L).\end{aligned} \tag{A-36}$$

Here, $P(k,L)$ is given by

$$P(k,L)=\sum_{i=1}^{L}h(k,i,L)H(i)\breve{H}B(i). \tag{A-37}$$

Subtracting $P(k,L-1)$ from $P(k,L)$ and using (A-3), it follows that

$$\begin{aligned}&P(k,L)-P(k,L-1)\\&=h(k,L,L)H(L)\breve{H}B(L)\\&+\sum_{i=1}^{L-1}(h(k,i,L)\\&-h(k,i,L-1))H(i)\breve{H}B(i)\\&=h(k,L,L)H(L)\breve{H}B(L)\\&+\sum_{i=1}^{L-1}(h(k,i,L)\\&-h(k,i,L-1))H(i)\breve{H}B(i)\\&=h(k,L,L)H(L)(\breve{H}B(L)\\&-\breve{H}\breve{\Phi}^L r_0(L-1)).\end{aligned} \tag{A-38}$$

Introducing

$$q(k,L)=P(k,L)(\breve{\Phi}^T)^L, \tag{A-39}$$

From (A-38) $q(k,L)$ satisfies

$$\begin{aligned}&q(k,L)=p(k,L-1)(\breve{\Phi}^T)^L\\&+h(k,L,L)H(L)(\breve{H}B(L)\\&-\breve{H}\breve{\Phi}^L r_0(L-1))(\breve{\Phi}^T)^L\\&=q(k,L-1)\breve{\Phi}^T\\&+h(k,L,L)H(L)(\breve{H}\breve{K}(L,L)\\&-\breve{H}\breve{\Phi}S_0(L-1)\breve{\Phi}^T).\end{aligned} \tag{A-40}$$

Hence, $h(k,L,L)$ satisfies

$$\begin{aligned}&h(k,L,L)R\\&=K_{\underline{x}\breve{x}}(k,k)(\breve{\Phi}^T)^{L-k}\breve{H}^T H^T(L)\\&-q(k,L)\breve{H}^T H^T(L).\end{aligned} \tag{A-41}$$

Substituting (A-40) into (A-41), after some manipulations, we obtain

$$\begin{aligned}&h(k,L,L)R\\&=(K_{\underline{x}\breve{x}}(k,k)(\breve{\Phi}^T)^{L-k}\breve{H}^T H^T(L)\\&-q(k,L-1)\breve{\Phi}^T\breve{H}^T H^T(L))\\&\times(R+H(k)\breve{H}(\breve{K}(k,k)\\&-\breve{\Phi}S_0(k-1)\breve{\Phi}^T)\breve{H}^T H^T(k))^{-1}.\end{aligned} \tag{A-42}$$

In the difference equation (A-40), from (A-13), (A-33), (A-37) and (A-39), the initial value $q(k,k)$ of $q(k,L)$ at $L=k$ is calculated by

$$q(k,k) = P(k,k)(\breve{\Phi}^T)^k$$
$$= \sum_{i=1}^{k} h\,(k,i,k)H(i)\breve{H}B(i)(\breve{\Phi}^T)^k$$
$$= \alpha(k)\sum_{i=1}^{k} J\,(i,k)H(i)\breve{H}B(i)(\breve{\Phi}^T)^k$$
$$= \Phi^k r(k)(\breve{\Phi}^T)^k$$
$$= S(k).$$

From (14) the fixed-point smoothing estimate of $\underline{x}(k)$ is given by

$$\hat{\underline{x}}(k,L) = \sum_{i=1}^{L} h\,(k,i,L)\breve{y}(i).$$

Subtracting $\hat{\underline{x}}(k,L-1)$ from $\hat{\underline{x}}(k,L)$, and using (A-3) and (A-30), it follows that

$$\hat{\underline{x}}(k,L) - \hat{\underline{x}}(k,L-1) = h(k,L,L)\breve{y}(L)$$
$$+ \sum_{i=1}^{L-1} (\,h(k,i,L) - h(k,i,L-1))\breve{y}(i)$$
$$= h(k,L,L)\breve{y}(L)$$
$$-h(k,L,L)H(L)\breve{H}\breve{\Phi}^L \sum_{i=1}^{L-1} J_0\,(i,L-1)\breve{y}(i)$$
$$= h(k,L,L)(\breve{y}(L) - H(L)\breve{H}\breve{\Phi}\hat{\breve{x}}(L-1,L-1)).$$

The initial value of $\hat{\underline{x}}(k,L)$ at $L=k$ is $\hat{\underline{x}}(k,k)$.

(Q.E.D.)

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