Variable Mesh Non Polynomial Spline Method for Singular Perturbation Problems Exhibiting Twin Boundary Layers

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Abstract: - In this paper, a variable mesh finite difference scheme using non polynomial spline is derived for the solution of singular perturbation problem with twin boundary layers. The equation of discretization for the problem is obtained by using the condition of continuity for the first order derivatives of the variable mesh non polynomial spline at the interior nodes. The discrete invariant imbedding algorithm is used to solve the tridiagonal system of the method. Convergence of the method is discussed and maximum absolute errors for the standard examples in comparison to the existing methods in the literature are presented to show the efficiency of the method.

Key-Words: - Singular perturbation problem, Non polynomial spline, Tridiagonal system, Truncation error

1 Introduction

We consider a second order singularly perturbed boundary value problem of the form:

$$\varepsilon y''(x) = p(x)y(x) + q(x)$$

with boundary conditions

$$y(0) = \gamma_1, \quad y(1) = \gamma_2$$

where γ_1, γ_2 are given constants, ε is a small positive parameter such that $0 < \varepsilon << 1$ and $p(x)$, $f(x)$ are small bounded real functions. It is known that the above problem exhibits boundary layers at both ends of the interval depends upon the properties of $p(x)$. These problems arise in many areas of engineering and applied mathematics. Examples of these are heat transport problem with Peclet numbers and Navier Stokes flows with large Reynold number. Because of the presence of boundary layers, difficulties are experienced in solving these types of problems using numerical methods with uniform mesh. In order to get a good approximation, a fine mesh is required in the boundary layer region. A wide variety of splines are described in [1 2]. The application of spline for the numerical solution of singularly perturbed boundary value problems has been described in many papers [3,4,5,6,7,8,9,10,11,12].

In this present paper, we have derived a variable mesh finite difference scheme using non polynomial spline for the solution of above problem. The main idea is to use the condition of continuity of the first order derivatives of the variable mesh non polynomial spline at the interior nodes as a discretization equation for the problem.

The paper is organized as follows: In section 2, the non polynomial spline for variable mesh is defined. In section 3, description of the numerical scheme is given. In section 4, convergence analysis of the proposed method is projected. Numerical illustrations are presented in section 5. Finally, conclusion is given in the last section.

2 Non Polynomial Spline for variable mesh

Let $0 = x_0 < x_1 < .... < x_{n-1} < x_n = 1$ be a sub division of an interval [0,1] with variable step size $h_i = x_i - x_{i-1}$ for i = 1 to n and $h_{i+1} = \sigma h_i$. Let $y(x)$ be the exact solution and y_i be an approximation to $y(x_i)$ obtained by the non polynomial cubic spline $S_i(x)$ passing through the points (x_i, y_i) and (x_{i+1}, y_{i+1}). We not only require that $S_i(x)$ satisfies interpolatory conditions at x_i and x_{i+1}, but also the continuity of first derivative at the common nodes (x_i, y_i) are fulfilled.

We write $S_i(x)$ in the form

$$S_i(x) = a_i + b_i(x - x_i) + c_i \sin\tau(x - x_i) + d_i \cos\tau(x - x_i), \quad i = 0,1,\ldots,n-1. \tag{1}$$

where a_i, b_i, c_i and d_i are constants and τ is a free parameter.

A non-polynomial function *S(x)* of class $C^2[a,\ b]$ interpolates *y(x)* at the grid points x_i for i = 0,1,...,*N* depends on a parameter τ and reduces to ordinary cubic spline *S(x)* in [*a*, *b*] as $\tau \to 0$.

To derive an expression for the coefficients of Eq. (1) in term of y_i, y_{i+1}, M_i and M_{i+1}, we first define

$$S_i(x_i) = y_i, S_i(x_{i+1}) = y_{i+1},$$
$$S''(x_i) = M_i, S''(x_{i+1}) = M_{i+1}.$$

From algebraic manipulation, we get the following expression for the unknowns:

$$a_i = y_i + \frac{M_i}{\tau^2},\ b_i = \frac{y_{i+1} - y_i}{h} + \frac{M_{i+1} - M_i}{\tau\theta},$$
$$c_i = \frac{M_i \cos\theta - M_{i+1}}{\tau^2 \sin\theta},\ d_i = -\frac{M_i}{\tau^2} \tag{2}$$

where $\theta = \tau\ h_{i+1}$, for i = 0,1,2,...,*n*-1.

Using the continuity of the first derivative at (x_i, y_i), i.e. $S'_{i-1}(x_i) = S'_i(x_i)$, we obtain the following relation for i =1,2,.......,*n*-1.

$$-(1+\sigma)y_i + \sigma\ y_{i-1} + y_{i+1} = h_{i+1}^2\left[\alpha_1 M_{i-1} + \beta\ M_i + \alpha_2 M_{i+1}\right] \tag{3}$$

where

$$\alpha_1 = \frac{-\sigma}{\theta^2} + \frac{\sigma}{\theta\sin\left(\frac{\theta}{\sigma}\right)},\quad \beta = \frac{\sigma}{\theta^2} + \frac{1}{\theta^2} - \frac{\cos\left(\frac{\theta}{\sigma}\right)}{\theta\sin\left(\frac{\theta}{\sigma}\right)} - \frac{\cos\theta}{\theta\sin\theta},$$

$$\alpha_2 = \frac{-1}{\theta^2} + \frac{1}{\theta\sin\theta} \text{ and } \theta = \tau,\ \ h_{i+1} = \tau\ \sigma\ h_i$$

The local truncation error $T_i(h_i)$ associated with our scheme (3) is

$$\begin{aligned} T_i(h_i) &= \left[-\frac{\sigma}{2}\left(1+\sigma\right) + \sigma^2\left(\alpha_1 + \beta + \alpha_2\right)\right] y_i^{(2)} h_i^2 \\ &+ \left[-\frac{\sigma}{6}\left(-1+\sigma^2\right) - \sigma^2\left(\alpha_1 - \sigma\alpha_2\right)\right] y_i^{(3)} h_i^3 \\ &+ \left[-\frac{\sigma}{24}\left(1+\sigma^3\right) + \frac{\sigma^2}{2}\left(\alpha_1 + \sigma^2\alpha_2\right)\right] y_i^{(4)} h_i^4 \\ &+ \left[-\frac{\sigma}{120}\left(-1+\sigma^4\right) - \frac{\sigma^2}{6}\left(\alpha_1 - \sigma^3\alpha_2\right)\right] y_i^{(5)} h_i^5 + 0\left(h^6\right) \end{aligned}$$

for the choice of

$$\alpha_1 = \frac{1+\sigma-\sigma^2}{12\sigma},\quad \alpha_2 = \frac{\sigma^2+\sigma-1}{12\sigma^2},\quad \beta = \frac{\sigma^3 + 4\sigma^2 + 4\sigma + 1}{12\sigma^2}.$$

The above scheme (3) indicates third order convergent and the truncation error will be

$$T_i(h_i) = \frac{1}{360}\left[\sigma\left(\sigma^2 - 1\right)\left(\sigma + 2\right)\left(2\ \sigma + 1\right)\right] y_i^{(5)} h_i^5 + O\left(h^6\right)$$

3 Numerical Method

We consider a second order singularly perturbed boundary value problem:

$$\varepsilon y''(x) = p(x)y(x) + q(x) \tag{4}$$
$$y(0) = \gamma_0,\quad y(1) = \gamma_1$$

where $0 < \varepsilon << 1$, $p(x)$, $q(x)$ are bounded continuous functions in (0, 1) and γ_1, γ_2 are finite constants. Substituting $M_j = y''_j$ in Eq. (4), we have

$$\varepsilon M_j = p(x_j)y(x_j) + q(x_j) \text{ for } j = i-1, i, \mathrm{i}+1$$

Substituting the above equations in Eq. (3), we get the following tridiagonal finite difference scheme:

$$\left(-\varepsilon\sigma + \alpha_1\ h_{i+1}^2\ p_{i-1}\right) y_{\mathrm{i-1}} + \left(\varepsilon\left(1+\sigma\right) + \beta\ h_{i+1}^2\ p_i\right) y_{\mathrm{i}}$$
$$+\left(-\varepsilon + \alpha_2\ h_{i+1}^2\ p_{i+1}\right) y_{\mathrm{i+1}} = -h_{\mathrm{i+1}}^2\left(\alpha_1\ q_{i-1} + \beta\ q_i + \alpha_2\ q_{i+1}\right)$$
$$\text{where } h_{\mathrm{i+1}} = \sigma\ h_i \tag{5}$$

We solve the tridiagonal system (5) by using the discrete invariant imbedding algorithm.

Remark: For $\varepsilon = 1$ (regular problem) $\sigma = 1$ (uniform mesh) we get

$$\left(-1+\frac{h^2}{12}p_{i-1}\right)y_{i-1}+\left(2+\frac{10h^2}{12}p_i\right)y_i$$
$$+\left(-1+\frac{h^2}{12}p_{i+1}\right)y_{i+1}=\frac{h^2}{12}\left(q_{i-1}+10\,q_i+q_{i+1}\right)$$

This is well known fourth order Numerov method for the regular problem

$$y''(x)=p(x)y(x)+q(x)$$

4 Convergence analysis

Writing the tridiagonal system of Eq. (5) in matrix-vector form, we get

$$AY=C \tag{6}$$

in which $A=\left(m_{i\,j}\right),\ 1\le i,j\le N\text{-}1$ is a tridiagonal matrix of order N-1 , with

$$m_{i\ i+1}=-\varepsilon+\alpha_2\ h_{i+1}^2\ p_{i+1}\ ,\quad i=1,2,...,N-2$$
$$m_{i\ i}=\varepsilon\left(1+\sigma\right)+\beta\ h_{i+1}^2\ p_i\ ,\quad i=1,2,...,N-1$$
$$m_{i\ i\text{-}1}=-\varepsilon\sigma+\alpha_1\ h_{i+1}^2\ p_{i-1}\ ,\quad i=2,3,...,N-1$$

and $C=\left(d_i\right)$; $1\le i\le N\text{-}1$ is a column vector with $d_i=-h_{i+1}^2\left(\alpha_1\ q_{i-1}+\beta\ q_i+\alpha_2\ q_{i+1}\right)$ and $Y=\left(y_1,y_2,...,y_{N-1}\right)^t$.

We also have $\quad A\overline{Y}-T(h)=C \qquad (7)$

where $\overline{Y}=\left(\overline{y_1},\overline{y_2},...,\overline{y_{N-1}}\right)^t$ is the actual solution and $T\left(h\right)=\left(T_1(h_1),T_2(h_2),....T_{N-1}(h_{N-1})\right)^t$ is the local truncation error with $T_i(h_i)$ given by

$$T_i(h_i)\ =\frac{\varepsilon}{360}\sigma\left(\sigma_i^2-1\right)\left(\sigma_i+2\right)\left(2\sigma_i+1\right)\ h_i^5y^{(5)}(x_i)+0\left(h_i^6\right) \tag{8}$$

From Eq. (6) and Eq. (7), we get

$$A\left(\overline{Y}-Y\right)=T(h) \tag{9}$$

Thus the error equation is

$$AE=T(h) \tag{10}$$

where $\quad E=\overline{Y}-Y=\left(e_1,e_2,...,e_{N-1}\right)^t$.

Clearly, we have

$$S_1=\sum_{j=1}^{N-1}m_{1\,j}=\varepsilon\sigma+h_2^2\left(\beta\ p_1+\alpha_2\ p_2\right)$$
$$S_i=\sum_{j=1}^{N-1}m_{i\,j}=h_{i+1}^2\left(\alpha_1\ p_{i-1}+\beta\ p_i+\alpha_3\ p_{i+1}\right)$$
$$=h_i^2\ B_i\ ,\ \ i=2\,,3,...,N-2$$

where $B_i=\ \sigma^2\ \ \left(\alpha_1\ p_{i-1}+\beta\ p_i+\alpha_2\ p_{i+1}\right)$

$$S_{N-1}=\sum_{j=1}^{N-1}m_{N-1\,j}=\varepsilon+h_N^2\left(\alpha_1\ p_{N-2}+\beta\ p_{N-1}\right)$$

Since $\sigma>0$, it can be seen that α_1,α_2 are positive if $1+\sigma\text{-}\sigma^2>0$, $\sigma^2+\sigma-1>0$

which implies $\quad \sigma<\frac{1}{2}+\frac{\sqrt{5}}{2}\cong 1.618$

and $\quad \sigma>\frac{\sqrt{5}}{2}-\frac{1}{2}\cong 0.618$

Thus, we obtain $\quad 0.618<\sigma<1.618.$

We can choose h sufficiently small so that the matrix A is irreducible and monotone. It follows that A^{-1} exists and its elements are non negative. Hence from Eq. (10) we get,

$$E=A^{-1}T(h) \tag{11}$$

Also from the theory of matrices we have

$$\sum_{i=1}^{N-1}\ \overline{m}_{k,i}\ \ S_i=1\ \ ,\quad k=1,2,...,N\text{-}1 \tag{12}$$

where $\overline{m}_{k,i}$ is $\left(k,i\right)$ element of the matrix A^{-1}. Therefore,

$$\sum_{i=1}^{N\text{-}1}\ \overline{m}_{k,i}\le\frac{1}{\min\limits_{1\le\ i\ \le N-1}S_i}=\frac{1}{h_i^2B_{i_o}}\le\frac{1}{h_i^2\left|B_{i_o}\right|} \tag{13}$$

for some i_0 between 1 and N-1.

From Eq. (8), Eq. (11) and Eq. (12), we get

$$e_j=\sum_{i=1}^{N-1}\ \overline{m}_{k.i}\ \ T_i(h),\quad j=1,2...,N\text{-}1 \tag{14}$$

which implies $\quad e_j\le\frac{Ph^3}{\left|B_{i_0}\right|}\ \ ,\quad j=1,2...,N\text{-}1$

where P is a constant independent of h.

Therefore, $\quad \left\|E_i\right\|=O\left(h_i^3\right)$

i.e., the method reduces to a third order convergent for non uniform mesh. In addition, if $\sigma = 1$ (uniform mesh) gives $T_i(h_i) = O(h^6)$.

Therefore, from Eq. (14) we have

$$\|E_i\| = O(h^4) \text{ where } h = \max h_i$$

i.e., our method reduces to a fourth order convergent for uniform mesh.

5 Numerical illustration

In order to test the viability of the proposed method, we applied it to two singular perturbation problems having a thin boundary layer of $O(\sqrt{\varepsilon})$ with two boundary layers. These examples have been chosen as they are widely discussed in literature and exact solutions are also available for comparison. We present maximum absolute errors with comparison and graphical solution to show the efficiency of the method.

Example 1. Consider the non-homogeneous singular perturbation problem

$$\varepsilon y''(x) - y(x) = \cos^2 \pi x + 2\varepsilon\pi^2 \cos 2\pi x;\ x \in [0,1]$$

with $y(0) = 0$ and $y(1) = 0$.
The exact solution is given by

$$y(x) = \frac{\left(e^{\left(-(1-x)/\sqrt{\varepsilon}\right)} + e^{\left(-x/\sqrt{\varepsilon}\right)}\right)}{\left(1 + e^{-1/\sqrt{\varepsilon}}\right)} - \cos^2 \pi x$$

The maximum absolute errors are presented in Table 1 for N = 16, 32, 64, 128, 256 with $\varepsilon = 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}$.

Example 2. Consider the non-homogeneous singular perturbation problem

$$-\varepsilon y'' + (1 + x) y = -40\left(x\left(x^2 - 1\right) - 2\varepsilon\right)$$

with $y(0) = 0$ and $y(1) = 0$.
The exact solution is given by $y(x) = 40x(1 - x)$

The maximum absolute errors with comparison are presented in Table 2 for different values of N = 16, 32 with $\varepsilon = 0.1 \times 10^{-3}, \ldots, 0.1 \times 10^{-9}$.

6 Conclusion

We have proposed a variable mesh non polynomial spline method for the solution of singular perturbation problems exhibiting twin layers. We have implemented the present method on standard test problems because they have been widely discussed in literature and exact solutions are also available for comparison. We have presented maximum absolute errors and compared the results with the existing methods to support the method. We have also presented graphical solution to the problems. The convergence analysis of the proposed method has been discussed. It is observed from the results that the present method approximate the exact solution very well for smaller value of ε also.

Table 1. Maximum absolute errors with comparison for Example 1

ε / N	2^4	2^5	2^6	2^7	2^8
Present method					
2^{-4}	4.07(-5)	2.53(-6)	1.58(-7)	9.87(-9)	6.17(-10)
2^{-5}	2.00(-5)	1.24(-6)	7.74(-8)	4.83(-9)	3.02(-10)
2^{-6}	5.45(-5)	3.42(-6)	2.14(-7)	1.34(-8)	8.39(-10)
2^{-7}	1.83(-4)	1.22(-5)	7.68(-7)	4.81(-8)	3.01(-9)
Surla and Stojanovic's method [11]					
2^{-4}	8.06(-3)	2.02(-3)	5.08(-4)	1.27(-4)	3.17(-5)
2^{-5}	7.11(-3)	1.79(-3)	4.48(-4)	1.12(-4)	2.80(-5)
2^{-6}	6.58(-3)	1.66(-3)	4.15(-4)	1.04(-4)	2.60(-5)
2^{-7}	6.36(-3)	1.61(-3)	4.03(-4)	1.01(-4)	2.52(-5)
Surla and Herceg and cvekovic's method [10]					
2^{-4}	4.14(-3)	1.02(-3)	2.54(-4)	6.35(-5)	1.58(-5)
2^{-5}	3.68(-3)	9.03(-4)	5.61(-5)	1.40(-5)	3.50(-6)
2^{-6}	3.45(-3)	8.40(-4)	2.08(-4)	5.20(-5)	1.30(-5)
2^{-7}	3.43(-3)	8.21(-4)	2.03(-4)	5.06(-5)	1.26(-5)
Surla and Vukoslavcevic's method [12]					
2^{-4}	1.20(-4)	7.47(-6)	4.67(-7)	2.90(-8)	4.39(-9)
2^{-5}	1.28(-4)	8.00(-6)	5.00(-7)	3.14(-8)	1.99(-9)
2^{-6}	1.60(-4)	1.00(-5)	6.26(-7)	3.92(-8)	2.31(-9)
2^{-7}	2.34(-4)	1.47(-5)	9.23(-7)	5.77(-8)	3.72(-9)
Kadalbajoo and Bawa's method [4]					
2^{-4}	7.09(-3)	1.77(-3)	4.45(-4)	1.11(-4)	2.78(-5)
2^{-5}	5.68(-3)	1.42(-3)	3.55(-4)	8.89(-5)	2.22(-5)
2^{-6}	4.07(-3)	1.01(-3)	2.54(-4)	6.35(-5)	1.58(-5)
2^{-7}	6.97(-3)	1.75(-3)	4.33(-4)	1.08(-4)	2.71(-5)

Table 2. Maximum absolute errors for Example 2

ε	Method in [10]	Method in [11]	Method in [12]	Present method
		N = 16		
0.1(-3)	2.5(-2)	2.6(-2)	6.5(-5)	3.55(-15)
0.1(-4)	2.1(-2)	2.4(-2)	3.6(-5)	1.77(-15)
0.1(-5)	7.0(-3)	1.7(-2)	3.3(-5)	1.77(-15)
0.1(-6)	7.5(-4)	6.9(-3)	2.6(-5)	3.55(-15)
0.1(-7)	7.4(-5)	2.3(-3)	2.0(-5)	3.55(-15)
0.1(-8)	6.7(-5)	7.6(-4)	2.0(-5)	1.77(-15)
0.1(-9)	0.0	2.4(-4)	1.1(-5)	3.55(-15)
		N = 32		
0.1(-3)	6.4(-3)	6.5(-3)	5.9(-5)	4.66(-15)
0.1(-4)	6.1(-3)	6.4(-3)	2.1(-5)	3.55(-15)
0.1(-5)	4.1(-3)	5.6(-3)	3.5(-5)	3.55(-15)
0.1(-6)	7.7(-4)	3.1(-3)	3.9(-5)	3.55(-15)
0.1(-7)	7.6(-5)	1.2(-3)	2.1(-5)	3.55(-15)
0.1(-8)	6.7(-6)	3.8(-4)	2.1(-5)	3.55(-15)
0.1(-9)	0.0	1.3(-4)	1.4(-5)	3.55(-15)

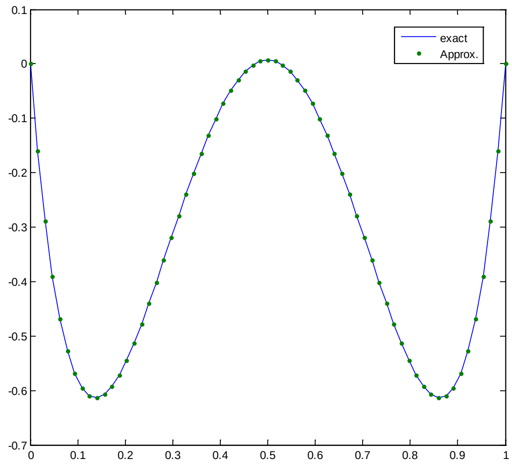

Fig 1. Numerical solution and exact solution of Example 1 with $N = 2^6$, $\varepsilon = 2^{-7}$.

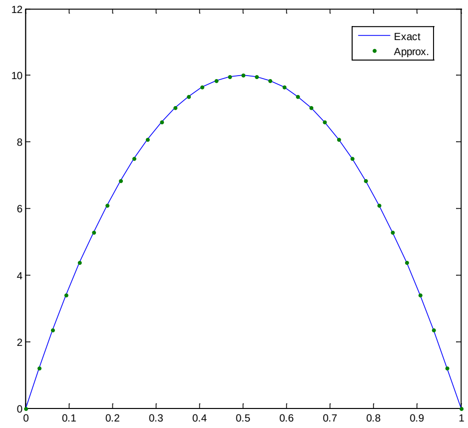

Fig 2. Numerical solution and exact solution of Example 3 with $N = 2^5$, $\varepsilon = 0.1\times10^{-3}$

References:

[1] J.H. Ahlberg, E.N. Nilson, J.H.Walsh, *The Theory of Splines and Their Applications*, Academic Press, New York, 1967.

[2] T.N.E. Greville, Theory and Application of Spline Functions, Academic Press, New York, 1969.

[3] M.K. Jain, T. Aziz, Numerical solution of stiff and convection–diffusion equation using adaptive spline function approximation, *Appl. Math. Model*. Vol. 7, No. 1, 1983, pp.57– 63.

[4] M.K. Kadalbajoo, R.K. Bawa, Variable-mesh difference scheme for singularly-perturbed boundary-value problem using splines, *J. Optimization Theory Appl*. Vol. 9, 1996, pp. 405–416.

[5] J.J.H. Miller, *On the convergence, uniformly in ε, of difference schemes for a two-point boundary singular perturbation problem, in Numerical analysis of singular perturbation problems*, Academic press, New York, 1979.

[6] K. Niijima, On a three-point difference scheme for a singular perturbation problem without a first derivative term I, *Mem. Numer. Math*., Vol. 7, 1980, pp. 1–10.

[7] K. Niijima, On a three-point difference scheme for a singular perturbation problem without a first derivative term II, *Mem. Numer. Math*., Vol. 7, 1980, pp. 11–27.

[8] J. Rashidinia, Variable mesh method for a singular two-point boundary value problem, *Int. J. Eng. Sci*., Vol. 14, 2003, pp. 23–33.

[9] J. Rashidinia, M. Ghasemi and Z. Mahmoodi, Spline approach to the solution of a singularly-perturbed boundary –value problems, *Applied Mathematics and Computation*, Vol. 189 2007, pp. 72-78.

[10] K. Surla, D. Herceg, L. Cvetkovic, A family of exponential spline difference scheme, *Review of research,* Faculty of science, Mathematics series. University of Novi sad. Vol. 19, 1991 pp. 12–23.

[11] K. Surla, M. Stojanovic, Solving singularly-perturbed boundary-value problems by splines in tension, *J. Comp. Appl. Math.* Vol. 24, No. 3, 1988, pp. 355–363.

[12] K. Surla, V. Vukoslavcevic, A spline difference scheme for boundary-value problems with a small parameter, *Review of research,* Faculty of science, Mathematics series. University of Novi sad. Vol. 25, No. 2, 1995, pp. 159–166.