Periodic traveling wave solutions for a coupled map lattice

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Abstract: A type of coupled map lattice (CML) is considered in this paper. What we want to do is to define the form of a traveling wave solution and to reveal its existence. Due to the infinite property of the problem, we have tried the periodic case, which can be dealt with on a finite set. The main approach for our study is the implicit existence theorem. The results indicate that if the parameters of the system satisfy some exact conditions, then there exists a periodic traveling wave solution in an exact neighborhood of a given one. However, these conditions are sufficient, but not necessary. In particular, the exact 2-periodic traveling wave solutions are also obtained. It gives some examples for the conditions of parameters, 2-periodic traveling wave solutions exist when these conditions are satisfied.

Key–Words: Coupled map lattice, Periodic traveling wave solution, Implicit existence theorem, Nagumo equation, Nontrivial solution

1 Introduction

Pattern dynamics in coupled map lattices (CMLs) have recently attracted considerable attention [1-9]. It has been found that CMLs exhibit a variety of space-time patterns such as kink-antikinks, traveling waves, space-time periodic structures, space-time intermittency and spatiotemporal chaos, etc. Furthermore, it is believed that CMLs possess the potential to explain phenomena associated with turbulence and other spatiotemporal systems. Among all space-time patterns, traveling waves play an important role in describing the long-term behavior of initial value problems in coupled map lattices (CMLs). And they also have their own practical background, such as, transition between different states of a physical system, propagation of patterns, and domain invasion of species in population biology. There have been many interesting studies on traveling waves in coupled map lattices (CMLs), see [10-20]. In particular, the periodic traveling wave solutions of coupled map lattices (CMLs) have also been researched in recent years. For example, Zhang et al. considered the existence of periodic traveling waves for the coupled logistic map lattice dynamical system in [15], and gave the conditions which can guarantee the existence of 2-periodic and 3-periodic traveling waves. In [16], Lin et al. observed the existence and nonexistence of doubly periodic traveling waves in cellular neural networks with polynomial reactions, they showed that the existence or nonexistence of doubly periodic traveling waves can be guaranteed by adjusting parameters of the networks.

In this paper we consider the following coupled map lattices (CMLs):

$$\begin{aligned} u_n^{t+1} &= u_n^t + \alpha\left(u_{n-1}^t - 2u_n^t + u_{n+1}^t\right) \\ &\quad +\beta f\left(u_n^t\right) \end{aligned} \tag{1}$$

where $t \in N = \{0, 1, 2, \cdots\}$ denotes the time and $n \in Z = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ denotes spatial coordinate, α is a positive constant, β is positive and is treated as a parameter, and

$$f(u) = u(u-a)(1-u), \ \ 0 < a < 1.$$

This is a discrete analogue of the well-known Nagumo equation of the form

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u), x \in R, t \in R^+, \tag{2}$$

where D is a positive constant. The continuous Nagumo equation (2) is used as a model for the spread of genetic traits [21] and for the propagation of nerve pulses in a nerve axon, neglecting recovery [22] and [23].

We note that equation (1) is also a discrete analogue of the following space discrete Nagumo equation

$$\frac{du_n}{dt} = d(u_{n-1} - 2u_n + u_{n+1}) + f(u_n), \tag{3}$$

where $n \in Z, d > 0$. Equation (3) has been used to derive equation (2), see [20].

In the recent studies of coupled map lattice (1), see [3-6, 10-20], particularly, by the Shen's viewpoint in [10], we know that equation (1) exists infinitely many stable non-monotone standing wave solutions connecting $u = 0$ and $u = 1$ when $0 < \alpha \ll 1$ and $\beta = 1$. At the same time, she also proved that (1) exists an unstable standing wave solution connecting $u = 0$ and $u = 1$ when $0 < \alpha \ll 1$ and $\beta = 1$. In this paper we consider the existence of periodic traveling wave solutions of problem (1) by using the implicit existence theorem. Accordingly, we also give out the exact lower bound of β. However, such lower bound is only a sufficient, but not necessary condition. In fact, we also obtain some exact 2-periodic traveling wave solutions by using the method similar to the one in [15], and find that the equation (1) has 2-periodic traveling wave solutions when β is sufficiently small. This is just the difference between present result with the one in [16], because according to Lin's view the equation (1) has no 2-periodic traveling wave solutions when $|\beta|$ is less than some positive number.

The present paper is structured as follows. In the next section we give the concept of periodic traveling wave solution. In Section 3, the existence of nontrivial periodic traveling wave solution of (1) is discussed, and some examples about the results of this work are presented. Finally we obtain some exact 2-periodic traveling wave solutions of (1).

2 Periodic Traveling Wave Solution

The solution $\{u_n^t\}_{n\in Z}^{t\in N}$ of problem (1) can be calculated by using the deduction method for given initial distribution $\{u_n^0\}_{n\in Z}$. In the following we use the simple denotation $\{u_n^t\}$ for $\{u_n^t\}_{n\in Z}^{t\in N}$. In this paper we search for a special kind of solution which satisfies $u_n^{t+1} = u_{n+l}^t$ for some $l \in Z$. Clearly, such a solution is called a **traveling wave** solution since in one period of time, the initial distribution is shifted l units to the left if l is positive, or l units to the right if l is negative. In the particular case when l is 0, there is no shift and the corresponding solution is also called a **stationary wave** solution. More generally, let us call a real double sequence $\{u_n^t\}_{n\in Z}^{t\in N}$ a traveling wave with 'velocity' $-l \in Z$ if $u_n^{t+1} = u_{n+l}^t$ for $t \in N$ and $n \in Z$.

Proposition 1 *The solution $\{u_n^t\}$ of problem (1) satisfies $u_n^{t+1} = u_{n+l}^t$ for some $l \in Z$ if and only if there exists a function $\varphi : Z \mapsto R$ such that*

$$u_n^t = \varphi(n + lt), \qquad n \in Z,\ t \in N. \tag{4}$$

Proof: If $u_n^t = \varphi(n + lt)$ holds true for some $l \in Z$ and $\varphi : Z \mapsto R$, then $u_n^{t+1} = \varphi(n + l(t + 1)) = \varphi((n + l) + lt) = u_{n+l}^t$ for all $n \in Z$ and $t \in N$. Conversely, if we let $\varphi(k) = u_k^0$ for every $k \in Z$, then

$$\begin{aligned} u_n^t &= u_{n+l}^{t-1} = u_{n+2l}^{t-2} = \cdots = u_{n+lt}^0 \\ &= \varphi(n + lt), \end{aligned}$$

and the proof is thus finished. □

A solution $\{u_n^t\}$ of problem (1) is said to be spatially periodic if there is a constant $\omega \in N$ such that $u_{n+\omega}^t = u_n^t$ for all $n \in Z$ and $t \in N$; and $\{u_n^t\}$ is said to be time periodic if there is a constant $\tau \in N$ such that $u_n^{t+\tau} = u_n^t$ for all $n \in Z$ and $t \in N$. Space-time periodic structures are important for understanding the phase transitions in CMLs, see [21].

We say a traveling wave solution $u_n^t = \varphi(n + lt)$ is periodic if there is a $\delta \in N$ such that $\varphi(m + \delta) = \varphi(m)$ for all $m = n + lt \in Z$. Notice that a periodic traveling wave propagate not only in space but also in time, it should be spatially periodic and time periodic. In fact, $\varphi((n + \delta) + lt) = \varphi(n + lt)$ implies $u_{n+\delta}^t = u_n^t$, which is ensured by Proposition 1. Similarly, we have $u_n^{t+\delta/l} = u_n^t$. So the period δ should be $\delta = kl$ for some $k \in N$.

3 Theory Results

Let

$$u_n^t = \varphi(n + lt), m = n + lt(l > 0), \tag{5}$$

substituting (5) into (1), we have

$$\begin{aligned} &\varphi(m + l) \\ &= \alpha[\varphi(m - 1) + \varphi(m + 1)] \\ &+ (1 - 2\alpha)\varphi(m) + \beta f(\varphi(m)). \end{aligned} \tag{6}$$

In view of Proposition 1, seeking the periodic traveling wave solutions of (1) is equivalent to seeking the periodic solution φ of (6). In the following we denote $\varphi_m = \varphi(m)$ for convenience and discuss the existence of periodic solution φ to equation (6) on the set

$$E = \{\varphi;\ \varphi_{m+\delta} = \varphi_m,\ \forall\ m \in Z\},$$

where the positive integer δ denotes the period of φ. For the case $\delta = 1$, we have $\varphi(m) \equiv \varphi(1)$ for all $m \in Z$, and it is a trivial case. So in the following we always assume $\delta > 1$.

Let $\lambda = 1/\beta$ then the equation (6) can be converted to

$$\begin{aligned} \lambda\varphi_{j+l} &= \alpha\lambda(\varphi_{j-1} + \varphi_{j+1}) \\ &\quad + (1 - 2\alpha)\lambda\varphi_j + f(\varphi_j). \end{aligned}$$

where $f(\varphi_j) = \varphi_j (\varphi_j - a)(1 - \varphi_j)$. To study the existence of the solution for the above equation, we define the functional:

$$\begin{aligned} & F_j(\Phi,\lambda) \\ = \; & -\lambda\varphi_{j+l} + \alpha\lambda(\varphi_{j-1} + \varphi_{j+1}) \\ & + (1-2\alpha)\lambda\varphi_j + f(\varphi_j), \end{aligned} \tag{7}$$

where $j = 1, 2, \cdots, \delta$, $\Phi = (\varphi_1, \varphi_2, \cdots, \varphi_\delta)^T$, where the superscript $'T'$ denotes the transpose and Φ is a column vector. It is easy to see the equation (1) has a periodic traveling wave solution $\varphi \in E$ only and only if $F_j(\Phi, \lambda) = 0$ for every $j = 1, 2, \cdots, \delta$.

It is easy to see that $\varphi^* \equiv 0, a$ and 1 are trivial periodic traveling solutions of equation (1), that is, $F_j(\Phi^*, \lambda) = 0$ for $j = 1, 2, \cdots, \delta$ and $\Phi^* = (\varphi^*, \varphi^*, \cdots, \varphi^*)^T$ with $\varphi^* \equiv 0, a$ or 1. In the following we discuss the existence of the nontrivial periodic traveling solutions $\varphi \in E$.

3.1 For the case $l = 1$

For the case $l = 1$, relation (7) becomes

$$\begin{aligned} & F_j(\Phi,\lambda) \\ = \; & \alpha\lambda\varphi_{j-1} + (\alpha-1)\lambda\varphi_{j+1} \\ & + (1-2\alpha)\lambda\varphi_j + f(\varphi_j) \end{aligned} \tag{8}$$

for $j = 1, 2, \cdots, \delta$. Since what we want to find is the periodic function $\varphi \in E$, the relations as follows will be used: $\varphi_{j-1} = \varphi_\delta$ for $j = 1$ and $\varphi_{j+1} = \varphi_1$ for $j = \delta$. Denote $F(\Phi, \lambda) = (F_1(\Phi, \lambda), F_2(\Phi, \lambda), \cdots, F_\delta(\Phi, \lambda))^T$, then its Frechet derivative about Φ acting on $\Psi = (\psi_1, \psi_2, \cdots, \psi_\delta)^T$ with $\psi \in E$ can be calculated as:

$$\frac{\partial F(\Phi,\lambda)}{\partial \Phi}\Psi = (a_{ij})_{\delta\times\delta}, \tag{9}$$

where

$a_{ij} = g(\varphi_i)\psi_i$ for $j = j = 1, \cdots, \delta$,
$a_{ij} = \alpha\lambda\psi_{i-1}$ for $j = i-1, i = 2, \cdots, \delta$,
$a_{ij} = (\alpha-1)\lambda\psi_{i+1}$ for $j = i+1, i = 1, \cdots, \delta-1$,
$a_{ij} = \alpha\lambda\psi_\delta$ for $j = \delta, i = 1$,
$a_{ij} = (\alpha-1)\lambda\psi_1$ for $j = 1, i = \delta$,
$a_{ij} = 0$ for the other,
$g(\varphi_j) = -3\varphi_j^2 + 2(a+1)\varphi_j - a + (1-2\alpha)\lambda$.

Correspondingly, we have

$$\frac{\partial F(\Phi^*,0)}{\partial \Phi}\Psi = (b_{ij})_{\delta\times\delta}, \tag{10}$$

where

$$b_{ij} = \begin{cases} g(\varphi_i^*)\psi_i, i = j = 1, \cdots, \delta, \\ 0, \text{ the other,} \end{cases}$$

and $\Phi^* = (\varphi_1^*, \varphi_2^*, \cdots, \varphi_\delta^*)^T$ is the trivial solution of $F(\Phi, 0) = 0$ and $g(\varphi_j^*) = -3(\varphi_j^*)^2 + 2(a+1)\varphi_j^* - a$ with $\varphi_j^* = 0, a$ or 1 for $j = 1, 2, \cdots, \delta$. Simple calculations show that

$$g(\varphi_j^*) = \begin{cases} -a & \text{for } \varphi_j^* = 0, \\ a(1-a) & \text{for } \varphi_j^* = a, \\ -(1-a) & \text{for } \varphi_j^* = 1. \end{cases} \tag{11}$$

We note that the trivial solution $\Phi^* = (\varphi_1^*, \varphi_2^*, \cdots, \varphi_\delta^*)^T$ of $F(\Phi, 0) = 0$ is the collection of $0, a$ and 1. The reason is, in case $\lambda = 0$ and $l = 1$ the equation (6) becomes $f(\varphi_j) = \varphi_j(\varphi_j - a)(1 - \varphi_j) = 0$, hence 0, a and 1 are all solutions to it. For example, in case $\delta = 4$, Φ^* can be $(1, a, 1, 0)^T$, $(0, 0, a, 1)^T$, etc.

We define the norm of a vector $\eta = (\eta_1, \eta_2, \cdots, \eta_\delta)^T$ by $\|\eta\| = \max\limits_{1\le j\le\delta} |\eta_j|$. Certainly R^δ is a Banach space under the above norm. Respectively, the norm of a metric $A = (a_{ij})_{\delta\times\delta}$ is defined as $\|A\| = \max\limits_{1\le i,j\le\delta} |a_{ij}|$.

To discuss the existence of solutions for equation (6), we take the approach given in [13,14] and consider it in the neighborhood of the trivial solution Φ^* by using the implicit function theorem. The following lemma given in [13] will be helpful.

Lemma 2 *Let X,Y be the Banach spaces and $\Lambda \subset R$ be the index set, $F \in C^1(X \times \Lambda, Y)$ with $F(x_0, \lambda_0) = 0$, $F_x^{-1}(x_0, \lambda_0) \in L(Y, X)$, $\|F_x^{-1}(x_0, \lambda_0)\| = M$, where M is a positive constant. Suppose the following conditions also hold true for some r and μ: (i) $\|F_x(x, \lambda) - F_x(x_0, \lambda_0)\| \le \frac{1}{2M}$ for $\|x - x_0\| \le r$ and $|\lambda - \lambda_0| \le \mu$; (ii) $\|F(x_0, \lambda)\| \le \frac{r}{2M}$ for $|\lambda - \lambda_0| \le \mu$. Then there exists exactly one function $T : B_\mu(\lambda_0) \mapsto B_r(x_0)$, $T(\lambda) = x_\lambda$, such that $x_{\lambda_0} = x_0$ and $F(x_\lambda, \lambda) = 0$, where $B_\mu(\lambda_0) = \{\lambda \in \Lambda;\ |\lambda - \lambda_0| \le \mu\}$ and $B_r(x_0) = \{x \in X;\ \|x - x_0\| \le r\}$.*

In the following we only consider the case $0 < a \le 1/2$. The case $1/2 < a < 1$ can be dealt with similarly. From (11) we have

$$\begin{cases} \max_{1\le j\le\delta} |g(\varphi_j^*)| = \max\{a, a(1-a), \\ \qquad 1-a\} = 1-a, \\ \min_{1\le j\le\delta} |g(\varphi_j^*)| = \min\{a, a(1-a), \\ \qquad 1-a\} = a(1-a). \end{cases} \tag{12}$$

Denote $\partial_\Phi F \equiv \partial_\Phi F/\partial\Phi$ and $\partial_\Phi F^{-1} \equiv \partial_\Phi F^{-1}/\partial\Phi$ for simple. We check the conditions for Lemma 2 as follows.

From (10) we infer that the operator $\partial_\Phi F(\Phi, 0)$ is invertible and

$$\partial_\Phi F^{-1}(\Phi^*, 0)\Psi = (c_{ij})_{\delta\times\delta}, \tag{13}$$

where

$$c_{ij}=\begin{cases}\frac{1}{g(\varphi_i^*)}\psi_i, i=j=1,\cdots,\delta,\\ 0, \text{the other.}\end{cases}$$

Accordingly,

$$\begin{aligned}&\left\|\partial_\Phi F^{-1}(\Phi^*,0)\right\|\\ =&\sup_{\|\Psi\|=1}\left\|\partial_\Phi F^{-1}(\Phi^*,0)\Psi\right\|\\ =&\sup_{\|\Psi\|=1}\max_{1\le j\le\delta}\left\{\frac{|\psi_j|}{|g(\varphi_j^*)|}\right\}\\ =&\frac{1}{a(1-a)},\end{aligned}\tag{14}$$

and hence $M=1/[a(1-a)]$.

On the other hand, notice that

$$\begin{aligned}&\max_{1\le j\le\delta}|(a+1)-3\varphi_j^*|\\ =&\max\{a+1,\ |a+1-3a|,\\ &|a+1-3|\}\\ =&2-a,\end{aligned}$$

and $\|\Phi-\Phi^*\|\le r$ implies $|\varphi_j-\varphi_j^*|\le r$ for $1\le j\le\delta$, we get

$$\begin{aligned}&|g(\varphi_j)-g(\varphi_j^*)|\\ \le&|(\varphi_j-\varphi_j^*)[-3(\varphi_j+\varphi_j^*)+2(a+1)]\\ &+(1-2\alpha)\lambda|\\ \le&|\varphi_j-\varphi_j^*|\cdot|3(\varphi_j^*-\varphi_j)-6\varphi_j^*\\ &+2(a+1)|+|1-2\alpha|\lambda\\ \le&3|\varphi_j-\varphi_j^*|^2+|2(a+1)\\ &-6\varphi_j^*|\cdot|\varphi_j-\varphi_j^*|+|1-2\alpha|\lambda\\ \le&3r^2+2(2-a)r+|1-2\alpha|\lambda.\end{aligned}\tag{15}$$

Notice that $|\lambda-0|\le\mu$, from (9), (10) and (15) it is easy to see

$$\begin{aligned}&\|\partial_\Phi F(\Phi,\lambda)-\partial_\Phi F(\Phi^*,0)\|\\ =&\sup_{\|\Psi\|=1}\|(\partial_\Phi F(\Phi,\lambda)-\partial_\Phi F(\Phi^*,0))\,\Psi\|\\ =&\sup_{\|\Psi\|=1}\max_{1\le j\le\delta}\left\{|g(\varphi_j)-g(\varphi_j^*)|\cdot|\psi_j|,\right.\\ &\left.|\alpha-1|\lambda|\psi_j|,\alpha\lambda|\psi_j|\right\}\\ =&\max_{1\le j\le\delta}\left\{|g(\varphi_j)-g(\varphi_j^*)|,|\alpha-1|\lambda,\alpha\lambda\right\}\\ =&\max\left\{\max_{1\le j\le\delta}|g(\varphi_j)-g(\varphi_j^*)|,\right.\\ &\left.|\alpha-1|\lambda,\alpha\lambda\right\}\\ \le&\max\left\{3r^2+2(2-a)r+|1-2\alpha|\mu,\right.\\ &\left.|\alpha-1|\mu,\ \alpha\mu\right\}.\end{aligned}\tag{16}$$

On the other hand, we have

$$\begin{aligned}&\|F(\Phi^*,\lambda)\|=\max_{1\le j\le\delta}|F_j(\Phi^*,\lambda)|\\ =&\max_{1\le j\le\delta}|\alpha\lambda\varphi_{j-1}^*+(\alpha-1)\lambda\varphi_{j+1}^*\\ &+(1-2\alpha)\,\lambda\varphi_j^*|\\ \le&\max_{1\le j\le\delta}|\alpha\varphi_{j-1}^*+(\alpha-1)\varphi_{j+1}^*\\ &+(1-2\alpha)\,\varphi_j^*|\mu,\end{aligned}\tag{17}$$

where $\varphi_j^*=0$, a or 1 for $1\le j\le\delta$ with $\varphi_0^*=\varphi_\delta^*$ and $\varphi_{\delta+1}^*=\varphi_1^*$. We search for the bounds of r and μ as follows.

Case 1: For the case $0<\alpha\le 1/2$, we have

$$\begin{aligned}&\max\left\{3r^2+2(2-a)r\right.\\ &\left.+|1-2\alpha|\mu,|\alpha-1|\mu,\alpha\mu\right\}\\ =&\max\left\{3r^2+2(2-a)r+(1-2\alpha)\mu,\right.\\ &\left.(1-\alpha)\mu\right\},\end{aligned}\tag{18}$$

$$\begin{aligned}&\max_{1\le j\le\delta}|\alpha\varphi_{j-1}^*+(\alpha-1)\varphi_{j+1}^*\\ &+(1-2\alpha)\,\varphi_j^*|\mu\\ =&|\alpha\varphi_{j-1}^*-(1-\alpha)\varphi_{j+1}^*\\ &+(1-2\alpha)\,\varphi_j^*|\mu\\ \le&(1-\alpha)\mu.\end{aligned}\tag{19}$$

In fact,

$$\begin{aligned}&\alpha\varphi_{j-1}^*-(1-\alpha)\varphi_{j+1}^*\\ &+(1-2\alpha)\,\varphi_j^*\\ \le&\alpha\cdot 1-(1-\alpha)\cdot 0+(1-2\alpha)\cdot 1\\ =&1-\alpha,\\ &\alpha\varphi_{j-1}^*-(1-\alpha)\varphi_{j+1}^*\\ &+(1-2\alpha)\,\varphi_j^*\\ \ge&\alpha\cdot 0-(1-\alpha)\cdot 1+(1-2\alpha)\cdot 0\\ =&-(1-\alpha).\end{aligned}$$

Let $\rho=1/2M=a(1-a)/2$. To satisfy $\|\partial_\Phi F(\Phi,\lambda)-\partial_\Phi F(\Phi^*,0)\|\le 1/2M$ and $\|F(\Phi^*,\lambda)\|\le r/2M$, from (18) and (19) it suffices for

$$\begin{cases}(1-\alpha)\mu\le\rho,\\ 3r^2+2(2-a)r+(1-2\alpha)\mu\le\rho,\\ (1-\alpha)\mu\le r\rho.\end{cases}\tag{20}$$

Choose $\mu=r\rho/(1-\alpha)$ and $r\le 1$ with

$$3r^2+2(2-a)r+\frac{1-2\alpha}{1-\alpha}\rho r=\rho,$$

then it yields

$$r = \frac{-A + \sqrt{A^2 + 12\rho}}{6}, \tag{21}$$

where $A = 2(2-a) + (1-2\alpha)\rho/(1-\alpha)$.

Case 2: For the case $1/2 < \alpha < 1$, we have

$$\begin{aligned} & \max\left\{3r^2 + 2(2-a)r \right. \\ & \left. +|1-2\alpha|\mu, |\alpha-1|\mu,\ \alpha\mu\right\} \\ = & \max\left\{3r^2 + 2(2-a)r \right. \\ & \left. +(2\alpha-1)\mu, \alpha\mu\right\}, \qquad (22) \\ & \max_{1\le j\le\delta} |\alpha\varphi_{j-1}^* + (\alpha-1)\varphi_{j+1}^* \\ & + (1-2\alpha)\,\varphi_j^*|\mu \\ = & |\alpha\varphi_{j-1}^* - (1-\alpha)\varphi_{j+1}^* \\ & - (2\alpha-1)\,\varphi_j^*| \\ \le & \alpha\mu. \qquad (23) \end{aligned}$$

Similarly, the sufficient conditions for $\|\partial_\Phi F(\Phi,\lambda) - \partial_\Phi F(\Phi^*, 0)\| \le 1/2M$ and $\|F(\Phi^*,\lambda)\| \le r/2M$ are as follows:

$$\begin{cases} \alpha\mu \le \rho, \\ 3r^2 + 2(2-a)r + (2\alpha-1)\mu \le \rho, \\ \alpha\mu \le r\rho. \end{cases} \tag{24}$$

Choose $\mu = r\rho/\alpha$ and $r \le 1$ with

$$3r^2 + 2(2-a)r + \frac{2\alpha-1}{\alpha}\rho r = \rho,$$

then it also yields (21) with $A = 2(2-a) + (2\alpha-1)\rho/\alpha$.

Case 3: For the case $\alpha \ge 1$, through the same process as the previous we get

$$\begin{aligned} r &= \frac{-A + \sqrt{A^2 + 12\rho}}{6}, \\ \mu &= \frac{\rho}{2\alpha - 1} r, \end{aligned} \tag{25}$$

where $A = 2(2-a) + \rho$.

So condition (i) for Lemma 2 holds for $\|\Phi - \Phi^*\| \le r$ and $|\lambda - 0| \le \mu$ respect to the above three cases. Hence the equation $F(\Phi,\lambda) = 0$ must have a solution $\Phi = \Phi(\lambda)$ which satisfies $\Phi(0) = \Phi^*$ and $F(\Phi(\lambda),\lambda) = 0$. This leads to our final results as follows.

Theorem 3 *For the case $0 < a \le 1/2$ and $l = 1$, if $0 < 1/\beta \le \mu$, then the equation (1) has a nontrivial periodic traveling wave solution $u_n^t = \varphi(n + lt) \in E$ in the neighborhood of Φ^*: $B_r(\Phi^*) = \{\Phi;\ \|\Phi - \Phi^*\| \le r\}$, where Φ^* is a solution of $F(\Phi^*, 0) = 0$, $r = (-A + \sqrt{A^2 + 12\rho})/6$ with $\rho = a(1-a)/2$, A and μ are given respect to the following cases: (a) $0 < \alpha \le 1/2$, $A = 2(2-a) + \rho\,(1-2\alpha)\,/\,(1-\alpha)$ and $\mu = \rho r/\,(1-\alpha)$; (b) $1/2 < \alpha < 1$, $A = 2(2-a) + \rho\,(2\alpha-1)\,/\alpha$, $\mu = \rho r/\alpha$; $\alpha \ge 1$, $A = 2(2-a) + \rho$, $\mu = \rho r\,(2\alpha-1)$.*

3.2 For the case $2 \le l < \delta$

For the case $2 \le l < \delta$ ($\delta = kl$ for some $k \in N$ with $\delta > 2$), relation (7) can be rewritten as:

$$\begin{aligned} & F_j(\Phi,\lambda) \\ = & -\lambda\varphi_{j+l} + \alpha\lambda(\varphi_{j-1} + \varphi_{j+1}) \\ & + (1-2\alpha)\,\lambda\varphi_j + f(\varphi_j) \end{aligned} \tag{26}$$

for $j = 1, 2, \cdots, \delta$, here $\varphi_0 = \varphi_\delta$ and $\varphi_{j+l} = \varphi_{j+l-\delta}$ for $j + l > \delta$. The Frechet derivative of $F(\Phi,\lambda)$ about Φ acting on $\Psi = (\psi_1, \psi_2, \cdots, \psi_\delta)^T$ with $\psi \in E$ can be calculated as the previous, and the matrix for $l = 2$ is

$$\frac{\partial F(\Phi,\lambda)}{\partial\Phi}\Psi = (d_{ij})_{\delta\times\delta}, \tag{27}$$

where

$d_{ij} = g(\varphi_i)\psi_i$ for $i = j = 1, \cdots, \delta$,
$d_{ij} = \alpha\lambda\psi_{i-1}$ for $j = i-1, i = 2, \cdots, \delta$,
$d_{ij} = \alpha\lambda\psi_{i+1}$ for $j = i+1, i = 1, \cdots, \delta-1$,
$d_{ij} = -\lambda\psi_{i+2}$ for $j = i+2, i = 1, \cdots, \delta-2$,
$d_{ij} = -\lambda\psi_{i+(2-\delta)}$ for $j = i + (2-\delta), i = \delta-1, \delta$,
$d_{ij} = \alpha\lambda\psi_1$ for $j = 1, i = \delta$,
$d_{ij} = \alpha\lambda\psi_\delta$ for $j = \delta, i = 1$,
$d_{ij} = 0$ for the other,

and $g(\varphi_j) = -3\varphi_j^2 + 2(a+1)\varphi_j - a + (1-2\alpha)\lambda$. In case $3 \le l < \delta$ we can get a similar matrix, the only difference is the locations of the terms $-\lambda\psi_{j+l}$ with $1 \le j \le \delta$. But this difference has no effect on the expression (10) for the case $\lambda = 0$. So (13)–(15) also hold. Similarly, we get

$$\begin{aligned} & \|\partial_\Phi F(\Phi,\lambda) - \partial_\Phi F(\Phi^*, 0)\| \\ = & \sup_{\|\Psi\|=1} \max_{1\le j\le\delta} \left\{|g(\varphi_j) - g(\varphi_j^*)| \cdot |\psi_j|, \right. \\ & \left. \alpha\lambda|\psi_j|, \lambda|\psi_j|\right\} \\ = & \max\left\{\max_{1\le j\le\delta} |g(\varphi_j) - g(\varphi_j^*)|, \right. \\ & \left. \alpha\lambda,\ \lambda\right\} \\ \le & \max\left\{3r^2 + 2(2-a)r + |1-2\alpha|\mu, \right. \\ & \left. \alpha\mu,\ \mu\right\}. \\ & \|F(\Phi^*,\lambda)\| \\ = & \max_{1\le j\le\delta} |\alpha\lambda(\varphi_{j-1}^* + \varphi_{j+1}^*) \\ & -\lambda\varphi_{j+l}^* + (1-2\alpha)\,\lambda\varphi_j^*| \end{aligned}$$

$$\begin{aligned} \leq & \max_{1\leq j\leq\delta} |\alpha(\varphi_{j-1}^* + \varphi_{j+1}^*) \\ & -\varphi_{j+l}^* + (1-2\alpha)\,\varphi_j^*|\mu, \end{aligned}$$

where $\varphi_j^* = 0$, a or 1 for $1 \leq j \leq \delta$. To satisfy $\|\partial_\Phi F(\Phi,\lambda) - \partial_\Phi F(\Phi^*,0)\| \leq 1/2M$ and $\|F(\Phi^*,\lambda)\| \leq r/2M$, it suffices for

$$r = \frac{-A + \sqrt{A^2 + 12\rho}}{6}, \tag{28}$$

where A and μ are given respect to the following two cases:

$$\begin{cases} A = 2(2-a) + (1-2\alpha)\rho, \\ \mu = \rho r, \quad 0 < \alpha \leq \frac{1}{2} \\ A = 2(2-a) + \left(1 - \frac{1}{2\alpha}\right)\rho, \\ \mu = \frac{\rho}{2\alpha} r. \qquad \alpha > \frac{1}{2}, \end{cases} \tag{29}$$

This leads to a result as follows.

Theorem 4 *For the case* $0 < a \leq 1/2$ *and* $2 \leq l < \delta$, *if* $0 < 1/\beta \leq \mu$, *then the equation (1) has a nontrivial periodic traveling wave solution* $u_n^t = \varphi(n + lt) \in E$ *in the neighborhood of* Φ^*: $B_r(\Phi^*) = \{\Phi;\ \|\Phi - \Phi^*\| \leq r\}$, *where* Φ^* *is a solution of* $F(\Phi^*, 0) = 0$, r *and* μ *are given by (28) and (29).*

3.3 For the case $l = \delta$

For the case $l = \delta$, relation (7) can be rewritten as:

$$\begin{aligned} F_j(\Phi,\lambda) &= \alpha\lambda(\varphi_{j-1} + \varphi_{j+1}) \\ & \quad -2\alpha\lambda\varphi_j + f(\varphi_j) \end{aligned}$$

for $j = 1, 2, \cdots, \delta$, here $\varphi_0 = \varphi_\delta$ and $\varphi_{\delta+1} = \varphi_1$. The Frechet derivative of $F(\Phi,\lambda)$ about Φ acting on $\Psi = (\psi_1, \psi_2, \cdots, \psi_\delta)^T$ with $\psi \in E$ can be calculated as

$$\frac{\partial F(\Phi,\lambda)}{\partial \Phi}\Psi = (e_{ij})_{\delta\times\delta},$$

where

$e_{ij} = g(\varphi_i)\psi_i$ for $i = j = 1, \cdots, \delta$,
$e_{ij} = \alpha\lambda\psi_{i-1}$ for $j = i-1, i = 2, \cdots, \delta$,
$e_{ij} = \alpha\lambda\psi_{i+1}$ for $j = i+1, i = 1, \cdots, \delta-1$,
$e_{ij} = a\lambda\psi_1$ for $j = 1, i = \delta$,
$e_{ij} = a\lambda\psi_\delta$ for $j = \delta, i = 1$,
$e_{ij} = 0$ for the other,
$g(\varphi_j) = -3\varphi_j^2 + 2(a+1)\varphi_j - a - 2\alpha\lambda$.

It is easy to see (10)–(14) also hold for this case. Similar deduction as for (15) reveals that $|g(\varphi_j) - g(\varphi_j^*)| \leq 3r^2 + 2(2-a)r + 2\alpha\lambda$. Hence

$$\begin{aligned} & \|\partial_\Phi F(\Phi,\lambda) - \partial_\Phi F(\Phi^*,0)\| \\ = & \sup_{\|\Psi\|=1} \max_{1\leq j\leq\delta} \left\{ |g(\varphi_j) - g(\varphi_j^*)| \cdot |\psi_j|, \right. \\ & \qquad \left. \alpha\lambda|\psi_j| \right\} \\ = & \max\left\{ \max_{1\leq j\leq\delta} |g(\varphi_j) - g(\varphi_j^*)|,\ \alpha\lambda \right\} \\ \leq & \max\left\{ 3r^2 + 2(2-a)r + 2\alpha\mu, \alpha\mu \right\}, \end{aligned}$$

$$\begin{aligned} & \|F(\Phi^*,\lambda)\| \\ = & \max_{1\leq j\leq\delta} |\alpha\lambda(\varphi_{j-1}^* + \varphi_{j+1}^*) - 2\alpha\lambda\varphi_j^*| \\ \leq & \max_{1\leq j\leq\delta} |\alpha(\varphi_{j-1}^* + \varphi_{j+1}^*) - 2\alpha\varphi_j^*|\mu \\ \leq & 2\alpha\mu, \end{aligned}$$

where $\varphi_j^* = 0$, a or 1 for $1 \leq j \leq \delta$. To satisfy $\|\partial_\Phi F(\Phi,\lambda) - \partial_\Phi F(\Phi^*,0)\| \leq 1/2M$ and $\|F(\Phi^*,\lambda)\| \leq r/2M$, it suffices for

$$\begin{cases} 3r^2 + 2(2-a)r + 2\alpha\mu \leq \rho, \\ \alpha\mu \leq \rho, \quad 2\alpha\mu \leq r\rho, \end{cases} \tag{30}$$

where $\rho = 1/2M$. Choose $\mu = r\rho/2\alpha$ and restrict $r \leq 1$, then the sufficient condition for (30) is

$$r = \frac{-A + \sqrt{A^2 + 12\rho}}{6}, \tag{31}$$

where $A = 2(2-a) + \rho$. This leads to the final result as follows.

Theorem 5 *For the case* $0 < a \leq 1/2$ *and* $l = \delta$, *if* $0 < 1/\beta \leq \mu$, *then the equation (1) has a nontrivial periodic traveling wave solution* $u_n^t = \varphi(n + lt) \in E$ *in the neighborhood of* Φ^*: $B_r(\Phi^*) = \{\Phi;\ \|\Phi - \Phi^*\| \leq r\}$, *where* Φ^* *is a solution of* $F(\Phi^*, 0) = 0$, $\mu = r\rho/2\alpha$ *and* r *is given by (31).*

3.4 Discussion

In the previous section, we have studied the existence of the periodic traveling wave solutions for the coupled map lattice equation (1) for the case $0 < a \leq 1/2$. In fact this request is coincident with that for the continuous model (2), in that the positive steady state $u = 1$ is dominant and there exists a wave front which connects $u = 0$ and $u = 1$. Respectively, we can get similar results for the case $1/2 < a < 1$ which reflects the steady state $u = 0$ is dominant. Since the CML problem (1) is an infinite one, it is very difficult to deal with, so we transform it to be an finite one and just discuss it on the set E. The main approach for our study is the implicit existence theorem. So all the results hold only in a local neighborhood of a given periodic traveling wave solutions Φ^*. In fact, what we have revealed is the exact bounds of the neighborhood and the parameters. Notice that a traveling wave solution $u_n^t = \varphi(n + lt)$ is periodic with period $\delta \in N$ means $\varphi(m + \delta) = \varphi(m)$ for all $m \in Z$, we mainly pay our attention to the case $1 \leq l \leq \delta$. In fact, δ should be k times of l for some $k \in N$.

We take the case $l = \delta = 4$ as an example to show the meaning of our results. In this case, the wave propagates 4 steps in the space x in a unit time t. Set $a = 1/3$, $\alpha = 1$ and $\Phi^* = (1, a, 1, 0)^T$ which reflects a easily known solution φ^* with $\varphi^*(i + 4k) = \varphi^*(i)$ for every $k \in N$ with $\varphi^*(1) = \varphi^*(3) = 1$, $\varphi^*(2) = a$ and $\varphi^*(4) = 0$. To ensure the existence of 4–periodic traveling wave solutions in a neighborhood of Φ^*, according to *Theorem 3*, it needs $\beta \geq 578.480524$ with the bound $r \approx 0.031116$. But we don't know whether there is a periodic traveling wave solution for $\beta < 578.480524$. It leaves to be an open problem.

We note that if the definition of the traveling wave solution is extended to be the form $u_n^t = \varphi(nk + lt)$ for some fixed $k, l \in N$ with $k, l > 1$, then there is no result *Proposition 1*. For example, $k = 2$ and $l = 3$ means the wave propagates 3 steps in the space x and in 2 units of time t. But how about a unit of time? That is, how does the wave propagate on the odd time points of the map lattice? This is also an open problem which attracts us to make further study.

4 About 2-Periodic Traveling Wave Solutions

In the above section, we have obtained some theory results. However, by the final discussion, we find that the theory results are not very good. Thus, in this section we hope to find some exact traveling wave solutions. It is well known that the 2-periodic solutions are important. Thus, we will seek 2-periodic solutions. In this case, we have two cases, that is, (A) l is odd, or (B) l is even.

When l is odd, the equation (6) can be reduced to

$$(2\alpha - 1)\varphi(m + 1) + (1 - 2\alpha)\varphi(m) + \beta f(\varphi(m)) = 0. \tag{32}$$

A simple method is to solve the algebraic system

$$\begin{cases} \gamma(x - y) + y(y - a)(1 - y) = 0, \\ \gamma(y - x) + x(x - a)(1 - x) = 0, \end{cases} \tag{33}$$

where

$$\gamma = \frac{2\alpha - 1}{\beta}.$$

However, it is difficult because we need to solve a 9-order algebraic equation.

In the following, we will choose an other method. Let

$$\varphi(m) = a_0 + a_1(-1)^m \ , \quad \varphi(m + 1) = a_0 - a_1(-1)^m \tag{34}$$

where $a_0, a_1 \in R$ and $a_1 \neq 0$. In this case, we have

$$-2\gamma a_1(-1)^m + f(\varphi(m)) = 0$$

which implies that

$$\begin{cases} a_1^2 = 2a_0(1 + a) - 3a_0^2 - (a + 2\gamma), \\ a_0^2 - aa_0 - a_0^3 + aa_0^2 \\ \qquad + (1 + a - 3a_0)a_1^2 = 0. \end{cases} \tag{35}$$

So a_0 satisfies the equation

$$x^3 - (1 + a)x^2 + \frac{1}{4}\left(1 + 3a + 3\gamma + a^2\right)x - \frac{1}{8}(a + 2\gamma)(a + 1) = 0 \tag{36}$$

or

$$x^3 + bx^2 + cx + d = 0, \tag{37}$$

where

$$\begin{aligned} b &= -(1 + a), \\ c &= \frac{1}{4}\left(1 + 3a + 3\gamma + a^2\right), \\ d &= -\frac{1}{8}(a + 2\gamma)(a + 1). \end{aligned}$$

Thus we need to solve the algebraic equation (37). To this end, let

$$x = y - \frac{b}{3},$$

(37) is changed into

$$y^3 + py + q = 0,$$

where

$$p = c - \frac{b^2}{3}, q = d - \frac{bc}{3} + \frac{2b^3}{27}.$$

Assume that

$$y = u + v,$$

we obtain the equation

$$(u + v)^3 + p(u + v) + q = 0$$

or

$$(u + v)(3uv + p) + u^3 + v^3 + q = 0.$$

Clearly,

$$u^3v^3 = -\frac{p^3}{27} \text{ and } u^3 + v^3 = -q$$

implies that the above equation holds. Thus, we have

$$u^3, v^3 = \frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

which implies that

$$\begin{cases} u_1 = A^{1/3}, v_1 = B^{1/3}, \\ u_2 = A^{1/3}\omega, v_2 = B^{1/3}\omega^2, \\ u_3 = A^{1/3}\omega^2, v_3 = B^{1/3}\omega, \end{cases}$$

$$\begin{cases} y_1 = A^{1/3} + B^{1/3}, \\ y_2 = A^{1/3}\omega + B^{1/3}\omega^2, \\ y_3 = A^{1/3}\omega^2 + B^{1/3}\omega, \end{cases}$$

or

$$\begin{cases} x_1 = A^{1/3} + B^{1/3} - \frac{b}{3}, \\ x_2 = A^{1/3}\omega + B^{1/3}\omega^2 - \frac{b}{3}, \\ x_3 = A^{1/3}\omega^2 + B^{1/3}\omega - \frac{b}{3}, \end{cases}$$

where

$$A = \frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}},$$
$$B = \frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

Denotes

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27}.$$

The following results are easily obtained.

Proposition 6 $\Delta > 0$ *implies that the equation (36) has only one real root*

$$x_1 = A^{1/3} + B^{1/3} - \frac{b}{3};$$

When $\Delta = 0$*, the equation (36) has three real roots and two of them are equal, that is*

$$x_1 = 2 \times (\frac{q}{2})^{\frac{1}{3}} - \frac{b}{3}, x_{2,3} = -(\frac{q}{2})^{\frac{1}{3}} - \frac{b}{3};$$

If $\Delta < 0$*, then the equation (36) has three real roots:*

$$\begin{aligned} x_1 &= 2\sqrt{-\frac{p}{3}}\cos\frac{\varphi}{3} - \frac{b}{3}, \\ x_2 &= 2\sqrt{-\frac{p}{3}}\cos(\frac{\varphi}{3} + \frac{2\pi}{3}) - \frac{b}{3}, \\ x_3 &= 2\sqrt{-\frac{p}{3}}\cos(\frac{\varphi}{3} + \frac{4\pi}{3}) - \frac{b}{3} \end{aligned}$$

where $\cos\varphi = -\frac{q}{2}(-\frac{p}{3})^{-\frac{3}{2}}, 0 < \varphi < \pi.$

To obtain the 2-periodic traveling wave solutions of (1), we need to ask that a_0 is a real root of (36). On the other hand, from (35) we also need to ask that

$$a_1^2 = 2a_0(1+a) - 3a_0^2 - (a + 2\gamma) > 0.$$

In this case, a_0 satisfies the condition

$$\begin{aligned} &\frac{1}{3}a - \frac{1}{3}\sqrt{a^2 - a - 6\gamma + 1} + \frac{1}{3} < a_0 \\ &< \frac{1}{3}a + \frac{1}{3}\sqrt{a^2 - a - 6\gamma + 1} + \frac{1}{3} \end{aligned} \tag{38}$$

when $a^2 - 6\gamma - a + 1 > 0$.

Proposition 7 *Assume that* $a^2 - 6\gamma - a + 1 > 0$ *and that* a_0 *is a real root of (36), which satisfies the condition (38). Then the equation (1) may exist a non-trivial 2-periodic traveling wave solution.*

When $a + 2\gamma = 0$, that is $a = 2(1-2\alpha)/\beta$ and $0 < \alpha < 1/2$, in view of Proposition 7, when a_0 satisfies the condition

$$0 < a_0 < \frac{2}{3}(1+a), \tag{39}$$

there may exists a non-trivial 2-periodic traveling wave solution.

In this case, the equation (36) is reduced to

$$x^3 - (1+a)x^2 + \frac{1}{4}\left(1 + \frac{3a}{2} + a^2\right)x = 0$$

which has the roots

$$x_{1,2} = \frac{a+1}{2} \pm \frac{\sqrt{2a}}{4} \text{ and } x_3 = 0.$$

Note that

$$a_1^2 = 2a_0(1+a) - 3a_0^2.$$

Thus, we can obtain a trivial solution $\varphi(m) = 0$ when $x_3 = 0$.

For

$$a_0 = x_1 = \frac{a+1}{2} + \frac{\sqrt{2a}}{4},$$

a_0 satisfies the condition (39) for $0 < a < 1/2$, we have

$$a_1 = \pm\sqrt{\frac{1}{4}(a^2 + \frac{1}{2}a + 1) - \frac{\sqrt{2a}}{4}(1+a)}.$$

For

$$a_0 = x_2 = \frac{a+1}{2} - \frac{\sqrt{2a}}{4},$$

when $0 < a < 1$, a_0 satisfies the condition (39), and we have

$$a_1 = \pm\sqrt{\frac{1}{4}(a^2 + \frac{1}{2}a + 1) + \frac{\sqrt{2a}}{4}(1+a)}.$$

So it is easy to see the following result.

Proposition 8 *When l is odd and*

$$a = \frac{2(1-2\alpha)}{\beta}, 0 < \alpha < \frac{1}{2} \tag{40}$$

then the equation (1) exists a 2-periodic traveling wave solution

$$u_n^t = a_0 + a_1 (-1)^{n+lt},$$

where

$$a_0 = \frac{a+1}{2} - \frac{\sqrt{2a}}{4}$$

and

$$a_1 = \pm\sqrt{\frac{1}{4}(a^2 + \frac{1}{2}a + 1) + \frac{\sqrt{2a}}{4}(1+a)}.$$

When l is odd and $a = 2(1-2\alpha)/\beta < 1/2$, the equation (1) exists another 2-periodic traveling wave solution

$$u_n^t = a_0 + a_1 (-1)^{n+lt},$$

where

$$a_0 = \frac{a+1}{2} + \frac{\sqrt{2a}}{4}$$

$$a_1 = \pm\sqrt{\frac{1}{4}(a^2 + \frac{1}{2}a + 1) - \frac{\sqrt{2a}}{4}(1+a)}.$$

Remark 9 *In Theorem 3 we can see that the equation (1) has a nontrivial periodic traveling wave solution when $1/\beta$ is bound. However, in Proposition 8, $1/\beta$ can be unbound when α approaches $1/2$ from the left. In such a case the equation (1) still have the nontrivial periodic traveling wave solution.*

For example, we set $l = 1, a = 0.5, \alpha = 0.4 < 1/2$, according to the Theorem 3, the equation (1) has a nontrivial periodic traveling wave solution when $\beta \geq 121.36$. But when $\beta = 0.8 < 121.36$, then $a + 2\gamma = 0$, the equation (1) still has periodic traveling wave solution:

$$\varphi(m) = 0.5 \pm 0.866 * (-1)^m.$$

So the Theorem 3 gives a sufficient, but not necessary condition for the existence of periodic traveling wave solution of (1).

When l is even, we have

$$2\alpha\varphi(m+1) - 2\alpha\varphi(m) + \beta f(\varphi(m)) = 0.$$

When $\varphi(m) = a_0 + a_1(-1)^m$ and $\varphi(m+1) = a_0 - a_1(-1)^m$, the above formula can be simplified

$$-2\gamma a_1(-1)^m + f(\varphi(m)) = 0,$$

where $\gamma = \alpha/\beta$. In this case, we can also obtain Propositions 6 and 7. However, we cannot obtain Proposition 8 because $a + 2\gamma > 0$. But we can obtain the numerical 2-periodic traveling wave solution by using mathematical software.

For example, let $l = 2, a = 0.5, \alpha = 0.2, \beta = 83$, $a + 2\gamma = 0.505 > 0$, there are three 2-periodic traveling wave solutions

$$\begin{aligned}\varphi_1(m) &= 0.5 \pm 0.495 * (-1)^m,\\ \varphi_2(m) &= 0.254 \pm 0.251 * (-1)^m,\\ \varphi_3(m) &= 0.746 \pm 0.251 * (-1)^m.\end{aligned}$$

5 Conclusion

In this paper, we have investigated the existence of the periodic traveling wave solutions for the coupled map lattice (1). The conditions for the existence of the periodic traveling wave solutions are obtained by using the implicit existence theorem, that is, the equation (1) for the case $0 < a \leq 1/2$ has a nontrivial periodic traveling wave solution when $1/\beta$ is bound. In particular, we give the exact 2-periodic traveling wave solutions and some examples for the conditions ensuring that 2-periodic traveling wave solutions exist. From these examples, we can see that the equation (1) still have the nontrivial 2-periodic traveling wave solution even though $1/\beta$ is unbound. So we conclude that the conditions obtained in section 3 are sufficient but not necessary.

Acknowledgements: This work was financially supported by Tianjin University of Commerce with the grant number of 090115.

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