New hybrid steepest descent algorithms for variational inequalities over the common fixed points set of infinite nonexpansive mappings

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Abstract: New hybrid steepest descent algorithms which are different from Yamada's hybrid steepest descent algorithms are proposed for solving variational inequalities defined on the common fixed points set of infinite nonexpansive mappings. As the extensions of our main results , algorithms are also given for solving variational inequalities defined on the common fixed points set of infinite κ-strict pseudo-contractions.

Key–Words: Hilbert space, fixed point, nonexpansive mapping, strict pseudo-contraction, variational inequality, hybrid steepest descent algorithm

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, let C be a nonempty closed convex subset of H, and let $F : C \to H$ be a nonlinear operator. We consider the problem of finding a point $x^* \in C$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall\ x \in C. \tag{1}$$

This is known as the variational inequality problem (i.e., $VI(C, F)$), which is introduced initially and studied by Stampacchia [1] in 1964. In recent years, variational inequality problems have been extended to study a large variety of problems arising in structural analysis, economics, optimization, operations research and engineering sciences, see [2-7] and the references therein.

Generally, F is assumed to be strongly monotone and Lipschitzian. Relative definitions are stated as below.

Let C be a nonempty closed and convex subset of a real Hilbert space H, $F : C \to H$ and $T : C \to C$, then

(i) F is called Lipschitzian on C, if there exists a positive constant L such that

$$\|Fx - Fy\| \leq L\|x - y\|, \quad \forall x, y \in C; \tag{2}$$

(ii) F is called boundedly Lipschitzian on C, if for each nonempty bounded subset B of C, there exists a positive constant κ_B depending only on the set B such that

$$\|Fx - Fy\| \leq \kappa_B\|x - y\|, \quad \forall x, y \in B; \tag{3}$$

(iii) F is said to be η-strongly monotone on C, if there exists a positive constant η such that

$$\langle Fx - Fy, x - y \rangle \geq \eta\|x - y\|^2, \ \forall\ x, y \in C; \tag{4}$$

(iv) T is said to be a κ-strict pseudo-contraction if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \quad \forall\, x, y \in C.$$

Specially, T is said to be nonexpansive on C if $\kappa = 0$. Obviously, the class of nonexpansive mappings is the proper subclass of strict pseudo-contractions.

The following lemma is known to us.

Lemma 1 *Assume that C is a nonempty closed convex subset of a real Hilbert space H, $F : C \to H$ is Lipschitzian and strongly monotone, then variational inequality (1) has a unique solution.*

Let $T : H \to H$ be a nonexpansive mapping with the set of fixed points $Fix(T) = \{x \in C : Tx = x\}$ and $F : H \to H$ is L- Lipschitzian and η- strong monotone. Yamada [8] studied the variational inequality problem $VI(Fix(T), F)$ and proposed a hybrid steepest descent algorithm:

$$x_{n+1} = (I - \mu\lambda_n F)Tx_n \tag{5}$$

and proved the strong convergence, where the sequence $\{\lambda_n\} \subset (0, 1)$ and $\mu \in (0, 2\eta/L^2)$.

Theorem 2 *(see [8]) Assume that H is a real Hilbert space, $T : H \to H$ is nonexpansive such that $Fix(T) \neq \emptyset$ and $F : H \to H$ is η-strongly monotone and L-Lipschitzian. Fix a constant $\mu \in (0, 2\eta/L^2)$. Assume also that the sequence $\{\lambda_n\} \subset (0, 1)$ satisfies the conditions:*

(i) $\lambda_n \to 0$ ($n \to \infty$);
(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;
(iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, or
$\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$.

Take $x_0 \in H$ arbitrarily and define $\{x_n\}$ by (5), then $\{x_n\}$ converges strongly to the unique solution of $VI(Fix(T), F)$.

Yamada also studied the variational inequality problem defined on the common fixed points set of finite nonexpansive mappings. Let H be a Hilbert space, T_n ($n = 1, \cdots, N$) are nonexpansive mappings from H into itself. $F : H \to H$ be η-strongly monotone and Lipschitzian. Assume $\bigcap_{n=1}^{N} Fix(T_n) = Fix(T_1T_2 \cdots T_N) = Fix(T_2 \cdots T_NT_1) = \cdots = Fix(T_N - 1 \cdots T_1T_N) = Fix(T_NT_N - 1 \cdots T_2T_1) \neq \emptyset$. For solving variational inequality problem $VI(\bigcap_{n=1}^{N} Fix(T_n), F)$, Yamada proposed the following cyclic algorithm:

$$x_1 = T_1x_0 - \lambda_0\mu F(T_1x_0)$$

$$x_2 = T_2x_1 - \lambda_1\mu F(T_2x_1)$$

$$\cdots$$

$$x_N = T_Nx_{N-1} - \lambda_{N-1}\mu F(T_Nx_{N-1}),$$

$$x_{N+1} = T_1x_N - \lambda_N\mu F(T_1x_N)$$

$$\cdots .$$

Indeed, the algorithm above can be rewritten as:

$$x_{n+1} = T_{[n+1]}x_n - \mu\lambda_n F(T_{[n+1]}x_n) \tag{6}$$

where $\mu \in (0, 2\eta/L^2)$, $\{\lambda_n\} \subset (0, 1)$ and $T_{[n]} = T_{nmodN}$, namely, $T_{[n]}$ is one of $T_1, T_2, \ldots, T_N$ circularly. Yamada got the following result:

Theorem 3 *(see [8]) If $\{\lambda_n\} \subset (0, 1)$ satisfies the conditions:*

(i) $\lambda_n \to 0$ ($n \to \infty$);
(ii) $\sum_{n=1}^{\infty} \lambda_n = \infty$;
(iii) $\sum_{n=1}^{\infty} |\lambda_{n+N} - \lambda_n| < \infty$, or
$\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$.

then the sequence $\{x_n\}$ generated by (6) converges strongly to the unique solution x^* of $VI(\bigcap_{n=1}^{N} Fix(T_n), F)$.

Let F be a boundedly Lipschitzian and strongly monotone operator and C be a closed convex subset of H. Songnian He and Hong-Kun Xu [9] obtained the following results:

Theorem 4 *(see [9]) Assume that $F : C \to H$ is boundedly Lipschitzian on C (i.e., for each bounded subset B of C, F is Lipschizian on B). Assume also that F is η-strongly monotone on C. Then variational inequality (1) has a unique solution $x^* \in C$ such that*

$$\|x^* - u\| \leq \frac{1}{\eta}\|Fu\|, \tag{7}$$

where $u \in C$ is an arbitrary fixed point.

Songnian He and Hong-Kun Xu [9] also proved that iterative algorithms can be devised to approximate this solution if F is a boundedly Lipschitzian and strongly monotone operator and C is the set of fixed points of a nonexpansive mapping. They invented a hybrid iterative algorithm:

$$x_{n+1} = Tx_n - \lambda_n\mu F(Tx_n), \quad n \geq 0. \tag{8}$$

Theorem 5 *(see [9]) Assume that $F : H \to H$ is $\eta-$strongly monotone and boundedly Lipschitzian. Fix an $x_0 \in C = Fix(T)$ arbitrarily and let $\hat{C}$ be the closed ball centered at x_0 and with radius $2\|Fx_0\|/\eta$ (i.e., $\hat{C} = S(x_0, 2\|Fx_0\|/\eta)$). Denote by $\hat{\kappa}$ the Lipschitz constant of F on $\hat{C}$, and take a constant μ satisfying $0 < \mu < \eta/\hat{\kappa}^2$. Assume a sequence $\{\lambda_n\}$ in the unit interval $(0, 1)$ satisfies the conditions:*

(i) $\lambda_n \to 0$ ($n \to \infty$);
(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;
(iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, or
$\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$.

Suppose that the sequence $\{x_n\}$ is generated by (8), then $\{x_n\}$ converges strongly to the unique solution x^* of $VI(Fix(T), F)$.

Songnian He and Xiao-lan Liang [10] considered $VI(C, F)$ when F is a boundedly Lipschitzian and strongly monotone operator and C is the set of fixed points of a strict pseudo-contraction $T : H \to H$. Fix a point $x_0 \in Fix(T)$ arbitrarily, set $\hat{C} = S(x_0, 2\|Fx_0\|/\eta)$. Denote by $\hat{L}$ the Lipschitz constant of F on $\hat{C}$. Fix the constant μ satisfying $0 < \mu < \eta/\hat{L}^2$. Assume also that the sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy $\kappa \leq \alpha_n \leq \alpha < 1$ for a constant $\alpha \in (0,\ 1)$ and $0 < \lambda_n < 1$ ($n \geq 0$) respectively. Let $T_{\alpha_n} = \alpha_n I + (1 - \alpha_n)T$ and $T^{\alpha_n,\lambda_n} = (I - \mu\lambda_n F)T_{\alpha_n}$, define $\{x_n\}$ by the scheme:

$$x_{n+1} = T^{\alpha_n,\lambda_n}x_n = (I - \mu\lambda_n F)T_{\alpha_n}x_n \ \ (n \geq 0). \tag{9}$$

Theorem 6 *(see [10]) If the sequences* $\{\lambda_n\}$ *and* $\{\alpha_n\}$ *satisfy the conditions:*

(i) $\lambda_n \to 0 \quad (n \to \infty)$;

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,
$\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, *or* $\lim_{n\to\infty} \frac{\lambda_{n-1}}{\lambda_n} = 1$,
$\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\lambda_n} = 0$;

then $\{x_n\}$ *generated by (9) converges strongly to the unique solution* x^* *of* $VI(Fix(T), F)$.

Songnian He and Xiao-lan Liang [10] also considered $VI(C, F)$ when F is a boundedly Lipschitzian and strongly monotone operator, C is the set of common fixed points of finite κ_i-strict pseudo-contractions $T_i : H \to H$ $(i = 1, \cdots, N)$. For such a C, they designed the following hybrid iterative algorithm:

For each $i = 1, \cdots, N$, let

$$T_{\alpha_i} = \alpha_i I + (1 - \alpha_i)T_i,$$

where the constant α_i such that $\kappa_i < \alpha_i < 1$. They defined the cyclic algorithm as follows:

$$x_1 = T_{\alpha_1} x_0 - \mu\lambda_0 F(T_{\alpha_1} x_0),$$

$$x_2 = T_{\alpha_2} x_1 - \mu\lambda_1 F(T_{\alpha_2} x_1),$$

$$\cdots$$

$$x_N = T_{\alpha_N} x_{N-1} - \mu\lambda_{N-1} F(T_{\alpha_N} x_{N-1}),$$

$$x_{N+1} = T_{\alpha_1} x_N - \mu\lambda_N F(T_{\alpha_1} x_N),$$

$$\cdots.$$

Indeed, the algorithm above can be rewritten as:

$$x_{n+1} = T_{\alpha_{[n+1]}} x_n - \mu\lambda_n F(T_{\alpha_{[n+1]}} x_n), \qquad (10)$$

where $T_{\alpha_{[n]}} = \alpha_{[n]} I + (1 - \alpha_{[n]})T_{[n]}$, $T_{[n]} = T_{n mod N}$, namely, $T_{[n]}$ is one of $T_1, T_2, \ldots, T_N$ circularly.

Theorem 7 *(see [10]) If* $\{\lambda_n\} \subset (0, 1)$ *satisfies the conditions:*

(i) $\lambda_n \to 0 \ (n \to \infty)$;

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(iii) $\sum_{n=0}^{\infty} |\lambda_{n+N} - \lambda_n| < \infty$, *or*
$\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$;

then the sequence $\{x_n\}$ *generated by (10) converges strongly to the unique solution* x^* *of* $VI(\bigcap_{n=1}^{N} Fix(T_n), F)$.

In this paper, motivated by the research above, we introduce general iterative algorithms for solving variational inequality problems $VI(C, F)$ where C is the set of common fixed points of infinite non-expansive mappings of $T_n : H \to H$ $(n = 1, 2, \cdots)$ with $\bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$ and $F : H \to H$ is a η-strongly monotone and Lipschitzian operator or is a η-strongly monotone and boundedly Lipschitzian operator, respectively. For the two cases of F, we will prove their strong convergence respectively. These algorithms are different from Yamada's hybrid steepest descent algorithms.

In order to deal with some problems involving the common fixed point set of infinite nonexpensive mappings , W-mapping is often used, see [11-16]. The W-mapping is defined by

$U_{n,n+1} = I,$
$U_{n,n} = \gamma_n T_n U_{n,n+1} + (1 - \gamma_n)I,$
$U_{n,n-1} = \gamma_{n-1} T_{n-1} U_{n,n} + (1 - \gamma_{n-1})I,$
$\vdots$
$U_{n,k} = \gamma_k T_k U_{n,k+1} + (1 - \gamma_k)I,$
$U_{n,k-1} = \gamma_{k-1} T_{k-1} U_{n,k} + (1 - \gamma_{k-1})I,$
$\vdots$
$U_{n,2} = \gamma_2 T_2 U_{n,3} + (1 - \gamma_2)I,$
$W_n = U_{n,1} = \gamma_1 T_1 U_{n,2} + (1 - \gamma_1)I,$

where $\{\gamma_i\}(i = 1, 2 \ldots)$ is a sequence of real number such that $0 < \gamma_i < 1$ and $\sum_{i=1}^{\infty} \gamma_i = 1$. Such a mapping W_n is called a W-mapping generated by T_1, T_2, $\ldots$, T_n and $\gamma_1, \gamma_2, \ldots, \gamma_n$.

Since W-mapping contains many composite operations of $\{T_n\}$, it is complicated and it needs large computational work. In this paper, we will adopt new method for solving fixed point problem defined on the common fixed points set of infinite nonexpansive mappings. If $\{x_k\}$ $(k = 1, 2, \ldots)$ is a bounded sequence of H and $\{\omega_k\} \subset (0, 1)$ such that $\sum_{k=1}^{\infty} \omega_k = 1$. It is easy to verify that $\sum_{k=1}^{\infty} \omega_k x_k$ is convergent. Let $L_n = \sum_{k=1}^{n} \frac{\omega_k}{S_n} T_k$ $(n = 1, 2, \ldots)$, where $S_n = \sum_{k=1}^{n} \omega_k$. We will replace W-mapping by L_n to solve fixed point problems defined on the common fixed points set of infinite non-expansive mappings. Because L_n doesn't contain many composite operations of $\{T_n\}$, it needs less computational work and it is simplistic and easy to realize.

In this paper, we define $\{x_n\}$ by the scheme:

$$x_{n+1} = \lambda_n (I - \mu F)x_n + (1 - \lambda_n)L_n x_n, \qquad (11)$$

where $\{\lambda_n\} \subset (0, 1)$ and μ is a constant. We will prove $\{x_n\}$ generated by (11) converges strongly to the unique solution x^* of $VI(\bigcap_{n=1}^{\infty} Fix(T_n), F)$ under some conditions.

We will use the notations:

- $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
- $\omega_w(x_n) = \{x : \exists \ x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

2 Preliminaries

In this section, some lemmas are given which are important to prove our main results. Lemma 8 and Lemma 10 are clearly to us.

Lemma 8 *Let H be a real Hilbert space. The following expressions hold.*
(i) $\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2-$

$$t(1-t)\|x-y\|^2,\ \ \forall x,y\in H, \forall t\in[0,1].$$

(ii) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle,\ \ \forall\ x,y\in H.$

Lemma 9 *(see[17,18])Assume $\{a_n\}$ is a sequence of non-negative real numbers satisfying*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n + \sigma_n,\ n=0,1,2\ldots. \tag{12}$$

where $\{\gamma_n\}_{n=0}^{\infty}\subset(0,1)$,$\{\delta_n\}_{n=0}^{\infty}$ and $\{\sigma_n\}_{n=0}^{\infty}$ satisfy the conditions:
(i)$\sum_{n=1}^{\infty}\gamma_n=\infty$,
(ii) $limsup_{n\to\infty}\delta_n\le 0$,
(iii)$\sum_{n=1}^{\infty}|\sigma_n|<\infty$,
then $\lim_{n\to\infty}a_n=0$.

Lemma 10 *Let H be a real Hilbert space, $F:H\to H$ be η-strongly monotone and L-Lipschitzian. Fix a constant μ satisfying $0<\mu<2\eta/L^2$, then $I-\mu F$ is a contraction with the contraction coefficient $1-\tau$, where $\tau=\frac{1}{2}\mu(2\eta-\mu L^2)$.*

Lemma 11 *(see [19]) Let C be a nonempty closed convex subset of a real Hilbert space H and $T:C\to C$ a nonexpansive mapping. If a sequence $\{x_n\}$ in C such that $x_n\rightharpoonup z$ and $(I-T)x_n\to y$, then $(I-T)z=y$.*

Lemma 12 *(see [16]) Assume that $T:H\to H$ is a κ-strict pseudo-contraction, and the constant α satisfies $\kappa\le\alpha<1$. Let*

$$T_\alpha=\alpha I+(1-\alpha)T, \tag{13}$$

then T_α is nonexpansive and $Fix(T_\alpha)=Fix(T)$.

Lemma 13 *Let H be a real Hilbert space, $\{x_k\}\subset H$ is bounded, then*

$$\|\sum_{k=1}^{\infty}\omega_k x_k\|^2=\sum_{k=1}^{\infty}\omega_k\|x_k\|^2-\sum_{1\le k<l<\infty}\omega_k\omega_l\|x_k-x_l\|^2. \tag{14}$$

Proof: For any positive integer m, let $S_m=\sum_{k=1}^{m}\omega_k$. From Lemma 8 (i) we can get that

$$\begin{aligned}
&\|\sum_{k=1}^{\infty}\omega_k x_k\|^2\\
=\ &\|\sum_{k=1}^{m}\omega_k x_k+\sum_{k=m+1}^{\infty}\omega_k x_k\|^2\\
=\ &\|\sum_{k=1}^{m}\omega_k x_k\|^2+2\langle\sum_{k=1}^{m}\omega_k x_k,\sum_{k=m+1}^{\infty}\omega_k x_k\rangle\\
&+\|\sum_{k=m+1}^{\infty}\omega_k x_k\|^2\\
=\ &\|S_m\sum_{k=1}^{m}\omega_k x_k/S_m\|^2+R_m\\
=\ &S_m^2[\sum_{k=1}^{m}\omega_k\|x_k\|^2/S_m-1/S_m^2\sum_{1\le k<l\le m}\omega_k\omega_l\|x_k\\
&-x_l\|^2]+R_m\\
=\ &S_m\sum_{k=1}^{m}\omega_k\|x_k\|^2-\sum_{1\le k<l\le m}\omega_k\omega_l\|x_k-x_l\|^2\\
&+R_m
\end{aligned}$$

where $R_m=2\langle\sum_{k=1}^{m}\omega_k x_k,\sum_{k=m+1}^{\infty}\omega_k x_k\rangle+\|\sum_{k=m+1}^{\infty}\omega_k x_k\|^2$. As $\sum_{k=1}^{\infty}\omega_k x_k$ is convergent, then $\lim_{m\to\infty}R_m=0$. Thus (14) follows from taking $m\to\infty$.

Lemma 14 *Let H be a real Hilbert space and $T_k:H\to H(k=1,2,\cdots)$ be all non-expansive mappings with $\bigcap_{k=1}^{\infty}Fix(T_k)\ne\emptyset$. Let $T=\sum_{k=1}^{\infty}\omega_kT_k$ $(k=1,2,\cdots)$, where $\{\omega_k\}\subset(0,1)$ such that $\sum_{k=1}^{\infty}\omega_k=1$. Then T is a non-expansive mapping and $Fix(\sum_{k=1}^{\infty}\omega_kT_k)=\cap_{k=1}^{\infty}Fix(T_k)$.*

Proof: Firstly, we have

$$\begin{aligned}
&\|Tx-Ty\|^2\\
=\ &\|\sum_{k=1}^{\infty}\omega_kT_kx-\sum_{k=1}^{\infty}\omega_kT_ky\|^2\\
=\ &\|\sum_{k=1}^{\infty}\omega_k(T_kx-T_ky)\|^2\\
\le\ &\sum_{k=1}^{\infty}\omega_k\|T_kx-T_ky\|^2\\
=\ &\|x-y\|^2\ \ \forall x,y\in H.
\end{aligned}$$

T is a nonexpansive mapping.

Secondly, it is obvious that $\cap_{k=1}^{\infty}Fix(T_k)\subset Fix(\sum_{k=1}^{\infty}\omega_kT_k)$. Now we prove the inverse inclusion $Fix(\sum_{k=1}^{\infty}\omega_kT_k)\subset\cap_{k=1}^{\infty}Fix(T_k)$. Taking a fixed point $u\in\cap_{k=1}^{\infty}Fix(T_k)$, for any $x\in Fix(\sum_{k=1}^{\infty}\omega_kT_k)$, we have form Lemma 13 that

$$\begin{aligned} & \|x-u\|^2 = \|\sum_{k=1}^{\infty}\omega_k T_k x - u\|^2 \\ = & \|\sum_{k=1}^{\infty}\omega_k (T_k x - T_k u)\|^2 \\ = & \sum_{k=1}^{\infty}\omega_k \|T_k x - T_k u\|^2 \\ & - \sum_{1\le k<l<\infty}\omega_k\omega_l\|T_k x - T_l x\|^2 \\ \le & \sum_{k=1}^{\infty}\omega_k\|x-u\|^2 - \sum_{1\le k<l<\infty}\omega_k\omega_l\|T_k x - T_l x\|^2 \\ = & \|x-u\|^2 - \sum_{1\le k<l<\infty}\omega_k\omega_l\|T_k x - T_l x\|^2. \end{aligned}$$

Hence

$$\sum_{1\le k<l<\infty}\omega_k\omega_l\|T_k x - T_l x\|^2 = 0.$$

This implies that $T_k x = T_l x$ hold for all $k, l = 1, 2, \cdots$. $T_l x = \sum_{k=1}^{\infty}\omega_k T_k x = x$ for every $l = 1, 2, \cdots$. Then we get $Fix(\sum_{k=1}^{\infty}\omega_k T_k) \subset \cap_{k=1}^{\infty} Fix(T_k)$.

Lemma 15 *Let H be a real Hilbert space and $T_k : H \to H (k = 1, 2, \cdots)$ be all non-expansive mappings with $\bigcap_{k=1}^{\infty} Fix(T_k) \neq \emptyset$. Let $T = \sum_{k=1}^{\infty}\omega_k T_k$ where $\{\omega_k\} \subset (0,1)$ such that $\sum_{k=1}^{\infty}\omega_k = 1$. Assume $L_n = \sum_{k=1}^{n}\omega_k T_k / S_n$, where $S_n = \sum_{k=1}^{n}\omega_k$. L_n uniformly converges to T in each bounded subset S in H.*

Proof: Notice S is bounded and $\{T_k\}$ are non-expansive mappings, so $M = \sup_{x\in S, k\ge 1}\|T_k x\| < \infty$. For all $x \in S$, we have

$$\begin{aligned} & \|L_n x - Tx\| \\ = & \|\sum_{k=1}^{n}\omega_k T_k x/S_n - \sum_{k=1}^{\infty}\omega_k T_k x\| \\ = & \|\sum_{k=1}^{n}(\omega_k - \omega_k S_n) T_k x/S_n - \sum_{k=n+1}^{\infty}\omega_k T_k x\| \\ \le & \|\sum_{k=1}^{n}(1-S_n)\omega_k T_k x/S_n\| + \|\sum_{k=n+1}^{\infty}\omega_k T_k x\| \\ \le & (1-S_n)/S_n \sum_{k=1}^{n}\omega_k\|T_k x\| + \sum_{k=n+1}^{\infty}\omega_k\|T_k x\|. \\ \le & M(1-S_n)/S_n + M\sum_{k=n+1}^{\infty}\omega_k. \end{aligned}$$

Observe that $(1-S_n)/S_n \to 0$ and $\sum_{k=n+1}^{\infty}\omega_k \to 0$ as $n \to \infty$, we obtain $\|L_n x - Tx\| \to 0$ $(n \to \infty)$.

Lemma 16 *(see[20]) Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$, then $z = P_C x$ (i.e., z is metric projection of x on C and z satisfies $\|x-z\| = \inf\{\|x-y\|; \forall y \in C\}$). if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \le 0 \quad \forall y \in C. \tag{15}$$

3 Main results

In this section , we always assume that $\{T_n\}$ $(n = 1, 2, \cdots)$ is a sequence of nonexpensive mappings from H to itself with $\bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$, $Tx = \sum_{n=1}^{\infty}\omega_n T_n x$ with $\{\omega_k\} \subset (0,1)$ such that $\sum_{k=1}^{\infty}\omega_k = 1$ and $L_n = \sum_{k=1}^{n}\omega_k T_k/S_n (n = 1, 2...)$ with $S_n = \sum_{k=1}^{n}\omega_k$. By Lemma 15, L_n uniformly converges to T on every bounded subset S in H. By Lemma 14, we obtain that $Fix(\sum_{k=1}^{\infty}\omega_k T_k) = \cap_{k=1}^{\infty} Fix(T_k)$, thus $VI(\bigcap_{n=1}^{\infty} Fix(T_n), F)$ is equivalent to VI$(Fix(T), F)$.

Now we consider $VI(\bigcap_{n=1}^{\infty} Fix(T_n), F)$, where $F : H \to H$ is η-strongly monotone and L-Lipschitzian. It follows from Lemma 1 that $VI(\bigcap_{n=1}^{\infty} Fix(T_n), F)$ has a unique solution x^* satisfying

$$\langle Fx^*, x - x^*\rangle \ge 0, \quad \forall\ x \in \bigcap_{n=1}^{\infty} Fix(T_n). \tag{16}$$

Our first result is as follow.

Theorem 17 *Assume that $F : H \to H$ is η-strongly monotone and L-Lipschitzian. Fix a constant $\mu \in (0, 2\eta/L^2)$, $\{\lambda_n\} \subset (0,1)$ satisfies the conditions:*
(i) $\lambda_n \to 0$ ($n \to \infty$);
(ii) $\sum_{n=0}^{\infty}\lambda_n = \infty$;
(iii)$\sum_{n=0}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$, or
$\lim_{n\to\infty}\frac{\lambda_n}{\lambda_{n+1}} = 1$.
Take $x_0 \in H$ arbitrarily and define $\{x_n\}$ by (11), then $\{x_n\}$ converges strongly to the unique solution of $VI(\bigcap_{n=1}^{\infty} Fix(T_n), F)$.

Proof: We will prove the result by three steps.
Step 1. Show that $\{x_n\}$ is bounded. By Lemma 10, $I - \mu F$ is a contraction with the coefficient $1 - \tau$, where $\tau = \frac{1}{2}\mu(2\eta - \mu L^2)$. Notice $x^* \in Fix(T)$ and $T : H \to H$ is nonexpansive, we have

$$\begin{aligned} & \|x_{n+1} - x^*\| \\ = & \|\lambda_n(I-\mu F)x_n + (1-\lambda_n)L_n x_n - x^*\| \\ = & \|\lambda_n[(I-\mu F)x_n - x^*] + (1-\lambda_n)(L_n x_n - x^*)\| \\ = & \|\lambda_n[(I-\mu F)x_n - x^*] + (1-\lambda_n)(L_n x_n - L_n x^*)\| \end{aligned}$$

$$\begin{aligned}
&\le \lambda_n\|(I-\mu F)x_n-x^*\|+(1-\lambda_n)\|L_nx_n-L_nx^*\|\\
&\le \lambda_n\|(I-\mu F)x_n-x^*\|+(1-\lambda_n)\|x_n-x^*\|\\
&= \lambda_n\|(I-\mu F)x_n-(I-\mu F)x^*-\mu Fx^*\|\\
&\quad +(1-\lambda_n)\|x_n-x^*\|\\
&\le \lambda_n\|(I-\mu F)x_n-(I-\mu F)x^*\|+\lambda_n\|\mu Fx^*\|\\
&\quad +(1-\lambda_n)\|x_n-x^*\|\\
&\le \lambda_n(1-\tau)\|x_n-x^*\|+\lambda_n\|\mu Fx^*\|+(1-\lambda_n)\\
&\quad \|x_n-x^*\|\\
&= (1-\lambda_n\tau)\|x_n-x^*\|+\lambda_n\mu\|Fx^*\|\\
&= (1-\lambda_n\tau)\|x_n-x^*\|+\lambda_n\mu\tau\|Fx^*\|/\tau\\
&\le \max\{\|x_n-x^*\|,\frac{\mu}{\tau}\|Fx^*\|\}.
\end{aligned}$$

By induction we get

$$\|x_{n+1}-x^*\| \quad \le \max\{\|x_0-x^*\|,\tfrac{\mu}{\tau}\|Fx^*\|\}.$$

Thus $\{x_n\}$ is bounded. Since $L_n (n=1,2...)$ is nonexpansive and F is L-Lipschitzian, we can get

$$\|L_nx_n-L_nx^*\| \quad \le \max\{\|x_0-x^*\|,\tfrac{\mu}{\tau}\|Fx^*\|\}$$

and

$$\|Fx_n-Fx^*\| \quad \le L\max\{\|x_0-x^*\|,\tfrac{\mu}{\tau}\|Fx^*\|\},$$

then $\{L_nx_n\}$ and $\{Fx_n\}$ are also bounded.

Step 2. Show that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

Firstly, we have from scheme (11) that

$$\begin{aligned}
&\|x_{n+1}-L_nx_n\|\\
&= \|\lambda_n(I-\mu F)x_n+(1-\lambda_n)L_nx_n-L_nx_n\|\\
&= \|\lambda_n(I-\mu F)x_n-\lambda_nL_nx_n\|\\
&\le \lambda_n\|(I-\mu F)x_n\|+\lambda_n\|L_nx_n\|.
\end{aligned}$$

By the condition (i) and $\{L_nx_n\}$ and $\{Fx_n\}$ are bounded , we obtain $\|x_{n+1}-L_nx_n\|\to 0\ (n\to\infty)$.

Secondly, we prove $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Let $M=\sup_n[\|(I-\mu F)x_n\|+\|L_nx_{n-1}\|]<\infty$, it concludes that

$$\begin{aligned}
&\|x_{n+1}-x_n\|\\
&= \|\lambda_n(I-\mu F)x_n+(1-\lambda_n)L_nx_n-\lambda_{n-1}(I-\\
&\quad \mu F)x_{n-1}-(1-\lambda_{n-1})L_{n-1}x_{n-1}\|\\
&= \|\lambda_n(I-\mu F)x_n-\lambda_n(I-\mu F)x_{n-1}+(1-\lambda_n)\\
&\quad (L_nx_n-L_nx_{n-1})+(\lambda_n-\lambda_{n-1})(I-\mu F)x_{n-1}\\
&\quad -(1-\lambda_{n-1})L_{n-1}x_{n-1}+(1-\lambda_n)L_nx_{n-1}\|\\
&\le \|\lambda_n(I-\mu F)x_n-\lambda_n(I-\mu F)x_{n-1}\|+(1-\lambda_n)\\
&\quad \|x_n-x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|(I-\mu F)x_{n-1}\|\\
&\quad +\|(1-\lambda_n)L_nx_{n-1}-(1-\lambda_{n-1})L_{n-1}x_{n-1}\|\\
&\le \lambda_n(1-\tau)\|x_n-x_{n-1}\|+(1-\lambda_n)\|x_n-x_{n-1}\|\\
&\quad +|\lambda_n-\lambda_{n-1}|(\|(I-\mu F)x_{n-1}\|)+\|(1-\lambda_n)\\
&\quad L_nx_{n-1}-(1-\lambda_{n-1})L_nx_{n-1}+(1-\lambda_{n-1})\\
&\quad L_nx_{n-1}-(1-\lambda_{n-1})L_{n-1}x_{n-1}\|\\
&\le (1-\lambda_n\tau)\|x_n-x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\\
&\quad (\|(I-\mu F)x_{n-1}\|+\|L_nx_{n-1}\|)\\
&\quad +(1-\lambda_n)\|L_nx_{n-1}-L_{n-1}x_{n-1}\|\\
&\le (1-\lambda_n\tau)\|x_n-x_{n-1}\|+|\lambda_n-\lambda_{n-1}|M\\
&\quad +(1-\lambda_n)\|L_nx_{n-1}-L_{n-1}x_{n-1}\|.
\end{aligned}$$

Notice that $\{x_n\}$is bounded again, there exists a constant $M_1\ge 0$ such that $\sup_{k,l\ge 1}\|T_kx_l\|\le M_1$, we have

$$\begin{aligned}
&\|L_nx_{n-1}-L_{n-1}x_{n-1}\|\\
&= \|\sum_{k=1}^{n}\omega_kT_kx_{n-1}/S_n-\sum_{k=1}^{n-1}\omega_kT_kx_{n-1}/S_{n-1}\|\\
&= \|\omega_nT_nx_{n-1}/S_n+\sum_{k=1}^{n-1}(-\omega_n)\omega_kT_kx_{n-1}/S_nS_{n-1}\|\\
&\le \|\omega_nT_nx_{n-1}/S_n\|+\sum_{k=1}^{n-1}\omega_n\omega_k/S_nS_{n-1}\|T_kx_{n-1}\|\\
&\le \omega_nM_1/S_n+\omega_nM_1/S_n\\
&= 2\omega_nM_1/S_n,
\end{aligned}$$

consequently,

$$\sum_{n=1}^{\infty}\|L_nx_{n-1}-L_{n-1}x_{n-1}\|\le 2M_1\sum_{n=1}^{\infty}\omega_n/S_n.$$

Since $\sum_{n=1}^{\infty}\omega_n/S_n$ is convergent, it is easy to see that $\sum_{n=1}^{\infty}\|L_nx_{n-1}-L_{n-1}x_{n-1}\|$ is also convergent. Using Lemma 9, it follows $\|x_{n+1}-x_n\|\to 0\ (n\to\infty)$. It is easy to have

$$\begin{aligned}
&\|x_n-Tx_n\|\\
&= \|x_n-x_{n+1}+x_{n+1}-L_nx_n+L_nx_n-Tx_n\|\\
&\le \|x_{n+1}-x_n\|+\|x_{n+1}-L_nx_n\|+\|L_nx_n-Tx_n\|.
\end{aligned}$$

By Lemma 15 , we obtain $\|x_n-Tx_n\|\to 0\ (n\to\infty)$. It follows from Lemma 11 that $\omega_w(x_n)\subset Fix(T)$.

Step 3. Show that $\lim_{n\to\infty}\|x_n-x^*\|=0$. By Lemma 8 (ii), we obtain

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&= \|\lambda_n(I-\mu F)x_n+(1-\lambda_n)L_nx_n-x^*\|^2\\
&= \|\lambda_n(I-\mu F)x_n-\lambda_nx^*+(1-\lambda_n)(L_nx_n-L_nx^*)\|^2\\
&= \|\lambda_n(I-\mu F)x_n-\lambda_n(I-\mu Fx^*)-\lambda_n\mu Fx^*\\
&\quad -(1-\lambda_n)(L_nx_n-L_nx^*)\|^2\\
&\le \|\lambda_n(I-\mu F)x_n-\lambda_n(I-\mu Fx^*)-(1-\lambda_n)
\end{aligned}$$

$$
\begin{aligned}
& (L_n x_n - L_n x^*)\|^2 + 2\langle -\lambda_n \mu F x^*, x_{n+1} - x^*\rangle \\
\le\ & \lambda_n (1-\tau)\|x_n - x^*\| + (1-\lambda_n)\|x_n - x^*)\| \\
& +2\langle -\lambda_n \mu F x^*, x_{n+1} - x^*\rangle \\
\le\ & (1-\tau\lambda_n)\|x_n - x^*\|^2 + 2\mu\lambda_n\langle -Fx^*, x_{n+1} - x^*\rangle.
\end{aligned}
$$

In fact, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that

$$\limsup_{n\to\infty}\langle -Fx^*, x_n - x^*\rangle = \lim_{j\to\infty}\langle -Fx^*, x_{n_j} - x^*\rangle.$$

Without loss of generality, we may further assume that $x_{n_j} \rightharpoonup \tilde{x} \in Fix(T)$. Since x^* is the unique solution of $VI(Fix(T), F)$, we obtain

$$
\begin{aligned}
\limsup_{n\to\infty}\langle -Fx^*, x_n - x^*\rangle &= \lim_{j\to\infty}\langle -Fx^*, x_{n_j} - x^*\rangle \\
&= -\langle Fx^*, \tilde{x} - x^*\rangle \le 0.
\end{aligned}
$$

Finally , We conclude that $\lim_{n\to\infty}\|x_n - x^*\| = 0$ from the conditions (i)-(iii) and Lemma 9 .

By Theorem 17 , we get one algorithm for finding the common fixed point with minimum norm of infinite nonexpansive mappings.

Corollary 18 *Let $\{x_{n+1}\}$ be determined by the scheme:*

$$x_{n+1} = \lambda_n \gamma x_n + (1-\lambda_n) L_n x_n \tag{17}$$

where $\gamma \in (-1, 1)$ and $\{\lambda_n\} \subset (0, 1)$ satisfies the conditions:

(i) $\lambda_n \to 0(\ n \to \infty)$;

(ii) $\sum_{n=0}^{\infty}\lambda_n = \infty$;

(iii) $\sum_{n=0}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$, or $\lim_{n\to\infty}\frac{\lambda_n}{\lambda_{n+1}} = 1$.

Then $x_n \to P_{\bigcap Fix(T_n)}0$.

Proof: Taking $F = I$ in (11) , we obtain $L = 1$ and $\eta = 1$. Fix a constant $\mu \in (0, 2\eta/L^2) = (0, 2)$, it concludes that $-1 < 1-\mu < 1$. Let $\gamma = 1-\mu$, then (11) was rewritten into (17). Using Theorem 17, $\{x_n\}$ converges strongly to the unique solution $x^\dagger$ of $VI(\bigcap_{n=1}^{\infty} Fix(T_n), F)$, that is

$$\langle x^\dagger, x - x^\dagger\rangle \ge 0, \quad \forall\ x \in \bigcap_{n=1}^{\infty} Fix(T_n). \tag{18}$$

Hence

$$\langle 0 - x^\dagger, x - x^\dagger\rangle \le 0, \quad \forall\, x \in \bigcap_{n=1}^{\infty} Fix(T_n). \tag{19}$$

Using Lemma 16 and (19), we have that $x^\dagger = P_{\bigcap Fix(T_n)}0$.

Now we turn to discussing $VI(\bigcap_{n=1}^{\infty} Fix(T_n), F)$ where F is a boundedly Lipschitzian and strongly monotone operator. Fix a point $x_0 \in \bigcap_{n=1}^{\infty} Fix(T_n)$ arbitrarily, set $\hat{C} = S(x_0, 2\|Fx_0\|/\eta)$. Denote by $\hat{L}$ the Lipschitz constant of F on $\hat{C}$. Fix the constant μ satisfying $0 < \mu < \eta/\hat{L}^2$. It follows from Theorem 4 that $VI(\bigcap_{n=1}^{\infty} Fix(T_n), F)$ has a unique solution x^*.

Our second main result is as follow.

Theorem 19 *Assume that $F : H \to H$ is a boundedly Lipschitzian and strongly monotone operator. Fix a point $x_0 \in \bigcap_{n=1}^{\infty} Fix(T_n)$ arbitrarily, set $\hat{C} = S(x_0, 2\|Fx_0\|/\eta)$. Denote by $\hat{L}$ the Lipschitz constant of F on $\hat{C}$. Fix a constant μ satisfying $0 < \mu < \eta/\hat{L}^2$. Suppose $\{\lambda_n\} \subset (0, 1)$ satisfies the conditions:*

(i) $\lambda_n \to 0(\ n \to \infty)$;

(ii) $\sum_{n=0}^{\infty}\lambda_n = \infty$;

(iii) $\sum_{n=0}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$, or $\lim_{n\to\infty}\frac{\lambda_n}{\lambda_{n+1}} = 1$.

Take $x_0 \in \bigcap_{n=1}^{\infty} Fix(T_n)$ arbitrarily and define $\{x_n\}$ by (11), then $\{x_n\}$ converges strongly to the unique solution of $VI(\bigcap_{n=1}^{\infty} Fix(T_n), F)$.

Proof: We will also divide the proof into three steps.

Step 1. We prove that $x_n \in \hat{C}$ for all $n \ge 0$ by induction. It is trivial that $x_0 \in \hat{C}$. Suppose we have proved $x_n \in \hat{C}$, that is

$$\|x_n - x_0\| \le 2\|Fx_0\|/\eta. \tag{20}$$

By Lemma 10, $I - \mu F$ is a contraction on $\hat{C}$ with the contraction coefficient $1 - \hat{\tau}$, where $\hat{\tau} = \frac{1}{2}\mu(2\eta - \mu\hat{L}^2)$. Notice $L_n : H \to H$ is nonexpansive, we have

$$
\begin{aligned}
& \|x_{n+1} - x_0\| \\
=\ & \|\lambda_n(I-\mu F)x_n + (1-\lambda_n)L_n x_n - x_0\| \\
=\ & \|\lambda_n[(I-\mu F)x_n - x_0] + (1-\lambda_n)(L_n x_n - x_0)\| \\
=\ & \|\lambda_n[(I-\mu F)x_n - x_0] + (1-\lambda_n)(L_n x_n - L_n x_0)\| \\
\le\ & \lambda_n\|(I-\mu F)x_n - x_0\| + (1-\lambda_n)\|L_n x_n - L_n x_0\| \\
\le\ & \lambda_n\|(I-\mu F)x_n - x_0\| + (1-\lambda_n)\|x_n - x_0\| \\
=\ & \lambda_n\|(I-\mu F)x_n - (I-\mu F)x_0 - \mu F x_0\| \\
& +(1-\lambda_n)\|x_n - x_0\| \\
\le\ & \lambda_n\|(I-\mu F)x_n - (I-\mu F)x_0\| + \lambda_n\|\mu F x_0\| \\
& +(1-\lambda_n)\|x_n - x_0\| \\
\le\ & \lambda_n(1-\hat{\tau})\|x_n - x_0\| + \lambda_n\|\mu F x_0\| + (1-\lambda_n) \\
& \|x_n - x_0\| \\
=\ & (1-\lambda_n\hat{\tau})\|x_n - x_0\| + \lambda_n\mu\|Fx_0\| \\
=\ & (1-\lambda_n\hat{\tau})\|x_n - x_0\| + \lambda_n\mu\hat{\tau}\|Fx_0\|/\hat{\tau} \\
\le\ & \max\{\|x_n - x_0\|, \frac{\mu}{\hat{\tau}}\|Fx_0\|\}. \\
\le\ & \max\{\frac{2}{\eta}, \frac{\mu}{\hat{\tau}}\}\|Fx_0\|.
\end{aligned}
$$

On the other hand, since $0 < \mu < \eta/\hat{L}^2$ and $\hat{\tau} = \frac{1}{2}\mu(2\eta - \mu\hat{L}^2)$, we get

$$\frac{\mu}{\tau} = \frac{\mu}{\frac{1}{2}\mu(2\eta - \mu\hat{L}^2)} = \frac{2}{\eta + (\eta - \mu\hat{L}^2)} \leq \frac{2}{\eta}$$

this implies that

$$\|x_{n+1} - x_0\| \leq 2\|Fx_0\|/\eta.$$

It implies that $x_{n+1} \in \hat{C}$. Therefore, $x_n \in \hat{C}$ for all $n \geq 0$ and $\{x_n\}$ is bounded.

Since L_n $(n = 1, 2...)$ is nonexpansive and F is L-Lipschitzian on $\hat{C}$, we get

$$\|L_nx_n - L_nx_0\| \quad \leq \|x_n - x_0\| \quad \leq 2\|Fx_0\|/\eta$$

and

$$\|Fx_n - Fx_0\| \quad \leq \hat{L}\|x_n - x_0\| \quad \leq 2\hat{L}\|Fx_0\|/\eta,$$

then $\{L_nx_n\}$ and $\{Fx_n\}$ are also bounded.

Step 2. Show that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Firstly, we have from scheme (11) that

$$\begin{aligned}
& \|x_{n+1} - L_nx_n\| \\
= & \|\lambda_n(I - \mu F)x_n + (1 - \lambda_n)L_nx_n - L_nx_n\| \\
= & \|\lambda_n(I - \mu F)x_n - \lambda_nL_nx_n\| \\
\leq & \lambda_n\|(I - \mu F)x_n\| + \lambda_n\|L_nx_n\|.
\end{aligned}$$

By the condition (i) and $\{L_nx_n\}$ and $\{Fx_n\}$ are bounded , we obtain $\|x_{n+1} - L_nx_n\| \to 0$ $(n \to \infty)$.

Secondly, we prove $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Let $M = \sup_n[\|(I - \mu F)x_n\| + \|L_nx_n\|] < \infty$, we can get

$$\begin{aligned}
& \|x_{n+1} - x_n\| \\
= & \|\lambda_n(I - \mu F)x_n + (1 - \lambda_n)L_nx_n - \lambda_{n-1}(I - \\
& \mu F)x_{n-1} - (1 - \lambda_{n-1})L_{n-1}x_{n-1}\| \\
= & \|\lambda_n(I - \mu F)x_n - \lambda_n(I - \mu F)x_{n-1} + (1 - \lambda_n) \\
& (L_nx_n - L_nx_{n-1}) + (\lambda_n - \lambda_{n-1})(I - \mu F)x_{n-1} \\
& -(1 - \lambda_{n-1})L_{n-1}x_{n-1} + (1 - \lambda_n)L_nx_{n-1}\| \\
\leq & \|\lambda_n(I - \mu F)x_n - \lambda_n(I - \mu F)x_{n-1}\| + (1 - \lambda_n) \\
& \|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|(I - \mu F)x_{n-1}\| \\
& +\|(1 - \lambda_n)L_nx_{n-1} - (1 - \lambda_{n-1})L_{n-1}x_{n-1}\| \\
\leq & \lambda_n(1 - \hat{\tau})\|x_n - x_{n-1}\| + (1 - \lambda_n)\|x_n - x_{n-1}\| \\
& +|\lambda_n - \lambda_{n-1}|(\|(I - \mu F)x_{n-1}\|) + \|(1 - \lambda_n) \\
& L_nx_{n-1} - (1 - \lambda_{n-1})L_nx_{n-1} + (1 - \lambda_{n-1}) \\
& L_nx_{n-1} - (1 - \lambda_{n-1})L_{n-1}x_{n-1}\| \\
\leq & (1 - \lambda_n\hat{\tau})\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}| \\
& (\|(I - \mu F)x_{n-1}\| + \|L_nx_{n-1}\|) \\
& +(1 - \lambda_n)\|L_nx_{n-1} - L_{n-1}x_{n-1}\| \\
\leq & (1 - \lambda_n\hat{\tau})\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|M \\
& +(1 - \lambda_n)\|L_nx_{n-1} - L_{n-1}x_{n-1}\|.
\end{aligned}$$

By the same proof in Theorem 17 , we conclude that $\|x_{n+1} - x_n\| \to 0$ $(n \to \infty)$. By triangle inequality , we obtain $\|x_n - Tx_n\| \to 0$ $(n \to \infty)$. It follows from Lemma 11 that $\omega_w(x_n) \subset Fix(T)$.

Step 3. Show that $\lim_{n\to\infty} \|x_n - x^*\| = 0$. By Lemma 8(ii), we obtain

$$\begin{aligned}
& \|x_{n+1} - x^*\|^2 \\
= & \|\lambda_n(I - \mu F)x_n + (1 - \lambda_n)L_nx_n - x^*\|^2 \\
= & \|\lambda_n(I - \mu F)x_n - \lambda_nx^* + (1 - \lambda_n)(L_nx_n - \\
& L_nx^*)\|^2 \\
= & \|\lambda_n(I - \mu F)x_n - \lambda_n(I - \mu Fx^*) - \lambda_n\mu Fx^* \\
& -(1 - \lambda_n)(L_nx_n - L_nx^*)\|^2 \\
\leq & \|\lambda_n(I - \mu F)x_n - \lambda_n(I - \mu Fx^*) - (1 - \lambda_n) \\
& (L_nx_n - L_nx^*)\|^2 + 2\langle -\lambda_n\mu Fx^*, x_{n+1} - x^*\rangle \\
\leq & \lambda_n(1 - \hat{\tau})\|x_n - x^*\| + (1 - \lambda_n)\|x_n - x^*)\| \\
& +2\langle -\lambda_n\mu Fx^*, x_{n+1} - x^*\rangle \\
\leq & (1 - \hat{\tau}\lambda_n)\|x_n - x^*\|^2 + 2\mu\lambda_n\langle -Fx^*, x_{n+1} - x^*\rangle.
\end{aligned}$$

By the same proof in Theorem 17, we conclude that $\lim_{n\to\infty} \|x_n - x^*\| = 0$.

Remark 20 *In the practical computation, we can get a common fixed point in $\bigcap_{i=1}^{\infty} Fix(T_i)$ as the initial iterative point by using any kind of algorithms, for example, the algorithm 17 in Corollary 18 .*

Now we apply the two results above to solve variational inequalities defined on the common fixed points set of infinite strict pseudo-contractions.

Let $\{T_i\}$ $(i = 1, 2, \cdots)$ is a sequence of κ_i-strict pseudo-contractions from H to itself with $\bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$. For any T_i , fix a constant α_i such that $\kappa_i < \alpha_i < 1$. Let $T_{\alpha_i} = \alpha_iI + (1 - \alpha_i)T_i$. From Lemma 12, T_{α_i} is a nonexpansive mapping and $Fix(T_{\alpha_i}) = Fix(T_i)$.

Let $\hat{T}x = \sum_{i=1}^{\infty} \omega_iT_{\alpha_i}x$, where ω_i satisfy $\omega_i > 0$ and $\sum_{i=1}^{\infty} \omega_i = 1$. By Lemma 14, we get $Fix(\sum_{k=1}^{\infty} \omega_kT_{\alpha_i}) = \bigcap_{i=1}^{\infty} Fix(T_{\alpha_i})$ and $\hat{T}$ is a nonexpansive mapping. It is easy to get $\bigcap_{i=1}^{\infty} Fix(T_i) = \bigcap_{i=1}^{\infty} Fix(T_{\alpha_i})$ and $\bigcap_{i=1}^{\infty} Fix(T_i) = Fix(\hat{T})$. Let $\hat{L}_n = \sum_{i=1}^{n} \frac{\omega_i}{S_n}T_{\alpha_i}$, where $S_n = \sum_{i=1}^{n} \omega_i$ $(n = 1, 2, \ldots)$. By Lemma 15, $\hat{L}_n$ uniformly converges to $\hat{T}$ on every bounded subset S in H. Thus $VI(\bigcap_{i=1}^{\infty} Fix(T_i), F)$ is equivalent to VI$(Fix(\hat{T}), F)$.

We define $\{x_n\}$ by the scheme:

$$x_{n+1} = \lambda_n(I - \mu F)x_n + (1 - \lambda_n)\hat{L}_nx_n. \quad (21)$$

By Theorem 17 and Theorem 19, we obtain the following two results respectively.

Theorem 21 *Assume that $F : H \to H$ is η-strongly monotone and L-Lipschitzian. Fix a constant $\mu \in (0, 2\eta/L^2)$, $\{\lambda_n\} \subset (0, 1)$ satisfies the conditions:*

(i) $\lambda_n \to 0$ $(n \to \infty)$;

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, or

$\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$.

Take $x_0 \in H$ arbitrarily and define $\{x_n\}$ by (21), then $\{x_n\}$ converges strongly to the unique solution of $VI(\bigcap_{n=1}^{\infty} Fix(T_n), F)$.

Theorem 22 *Assume that $F : H \to H$ is a boundedly Lipschitzian and strongly monotone operator. Fix a point $x_0 \in \bigcap_{n=1}^{\infty} Fix(T_n)$ arbitrarily, set $\hat{C} = S(x_0, 2\|Fx_0\|/\eta)$. Denote by $\hat{L}$ the Lipschitz constant of F on $\hat{C}$. Fix the constant μ satisfying $0 < \mu < \eta/\hat{L}^2$.Suppose $\{\lambda_n\} \subset (0, 1)$ satisfies the conditions:*

(i) $\lambda_n \to 0$ ($n \to \infty$);

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, or

$\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$.

Take $x_0 \in \bigcap_{n=1}^{\infty} Fix(T_n)$ arbitrarily and define $\{x_n\}$ by (21), then $\{x_n\}$ converges strongly to the unique solution of $VI(\bigcap_{n=1}^{\infty} Fix(T_n), F)$.

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