The Singular Diffusion Equation with Boundary Degeneracy

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Abstract: For the heat conduction on a bounded domain with boundary degeneracy, though its diffusion coefficient vanishes on the boundary, it is still possible that the heat flux may transfer across the boundary. A known result shows that the key role is the ratio of the diffusion coefficient near the boundary. If this ratio is large enough, the heat flux transference has not any relation to the boundary condition but is completely controlled by the initial value. This phenomena shows there are some essential differences between the heat flux with boundary degeneracy and that without boundary degeneracy. However, under the assumption on the uniqueness of the weak solutions, the paper obtains that the weak solution of the singular diffusion equation with boundary degeneracy, has the same regular properties as the solution of a singular diffusion equation without boundary degeneracy.

Key–Words: Boundary degeneracy, Diffusion equation, Uniqueness, Regular property

1 Introduction

In this paper, we study the singular diffusion equation with boundary degeneracy

$$\frac{\partial u}{\partial t} = \operatorname{div}(d^{\alpha} \cdot |\nabla u|^{p-2}\nabla u), \tag{1}$$

where $(x,t) \in Q_T = \Omega \times (0,T)$, $\Omega \subset R^N$ is a bounded domain with appropriately smooth boundary, $p > 1, \alpha > 0$, and $d = d(x) = \operatorname{dist}(x, \partial\Omega)$. If $\alpha = 0$, then the equation (1) becomes the following classical evolutionary $p-$ Laplacian equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\nabla u). \tag{2}$$

This equation reflects the more practical process of heat conduction than the classical heat conduction equation $u_t = \Delta u$ does. For example, when $p > 2$, the solution of the equation (2) may possess the properties of finite speed of propagation, while $u_t = \Delta u$ always has the properties of infinite speed of propagation and it seems clearly contrary to the practice. There are a tremendous amount of related works for equation (2), one refers to the reference [8]-[13] and the books [1], [6], [7] etc and the reference therein. The authors also had done some research for this equation in [14]-[16]. However, unlike the equation (2), there is few reference related to the equation (1) except [2]-[3]. The works [2]-[3] only studied the well-posedness of the equation (1), while the properties of the corresponding weak solutions, such as the Harnack inequality, the large time behavior etc., are still undone.

For the equation (1), the diffusion coefficient depends on the distance function to the boundary. Since the diffusion coefficient vanishes on the boundary, it seems that there is no heat flux across the boundary. However the reference [2] shows that the fact might not coincide with what we image. In fact, the exponent α, which characterizes the vanishing ratio of the diffusion coefficient near the boundary, does determine the behavior of the heat transfer near the boundary. Let us give the definition of weak solution for equation (1) as follows:

Definition 1 *If the function $u(x,t)$ satisfies*

$$u \in C(0,T; L^2(\Omega)) \cap L^{\infty}(Q_T),$$
$$\tfrac{\partial u}{\partial t} \in L^2(Q_T),$$
$$d^{\alpha}|\nabla u|^p \in L^1(Q_T),$$

and for any test function $\varphi \in C_0^{\infty}(Q_T)$, the following integral equality holds

$$\int_{Q_T} (\frac{\partial u}{\partial t}\varphi + d^{\alpha} \cdot |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi)dxdt = 0. \tag{3}$$

then the function $u(x,t)$ is said to be a weak solution of the equation (1).

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According to reference [2], if $0 < \alpha < p - 1$, we can impose the Dirichlet boundary condition as usual

$$u(x,t) = g(x,t), (x,t) \in \partial\Omega \times (0,T). \tag{4}$$

Here and in what follows, we always assume that $g(x,t)$ is a function which can be extended to $\overline{Q_T}$ and the extended function is appropriately smooth. If $\alpha \geq p - 1$, then the heat conduction of equation (1) is entirely free from the limitations of the boundary condition. In other words, the problem of heat conduction is entirely controlled by the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{5}$$

In details, the reference [2] had got the following important propositions.

Proposition 2 *If $0 < \alpha < p - 1$, then for any u_0 satisfying*

$$u_0(x) \in L^\infty(\Omega), \quad d^\alpha |\nabla u_0|^p \in L^1(\Omega) \tag{6}$$

there exists at least one weak solution of the initial boundary problem (1)-(4)-(5). Moreover, the solution of the problem is unique.

Proposition 3 *If $\alpha \geq p - 1$, then the equation (1) admits at most one weak solution with initial value u_0, no matter what the boundary value is. Moreover, for any u_0 as in proposition 2, there exists at least one weak solution of the equation (1) with initial value (5).*

These two propositions show that there are some essential differences between the equation (1) and the equation (2).

In this paper, we will discuss the regularity of the solution for the equation (1). If the equation (1) exists unique a solution, we will show that its weak solution has the same regular properties as that of the solution for the equation (2). In turn, it shows that, although the diffusion coefficient degenerating on boundary has directly impact on the heat flow cross-border situation, in terms of smoothness of solutions, whether the diffusion coefficient degenerating on the boundary or not does not play an essential role.

According to [4], the distance function is always almost everywhere derivable and $|\nabla d| = 1$ is true in the distribution sense. In what follows, for simplicity, we assume that the distance function $d(x) = \mathrm{dist}(x, \partial\Omega)$ is a derivable function for $x \in \Omega$.

2 The gradient bounded properties of the solution

Before to prove the theorem, we will introduce the following lemmas from reference [1].

Lemma 4 *There exists a constant γ only depending on p, q, N such that for any $v \in V_0^{q,p}(\Omega_T)$ and $h = \frac{p(q+N)}{N}$,*

$$\int\int_{\Omega_T} |v(x,t)|^h dxdt \leq \gamma \left(\int\int_{\Omega_T} |\nabla v(x,t)|^p dxdt \right) \cdot \left(\operatorname{ess\,sup} \int_\Omega |v(x,t)|^q dx \right)^{\frac{p}{N}},$$

where $V_0^{q,p}(\Omega_T)$ is the closure of $C_0^\infty(\Omega_T)$ in space of $V^{q,p}(\Omega_T)$, and

$$V^{q,p}(\Omega_T) = L^\infty(0,T;L^q(\Omega)) \bigcap L^p(0,T;W^{1,p}(\Omega)).$$

Lemma 5 *Let $Q_n (n = 1, 2, \cdots)$ be a sequence of bounded open sets in Ω_T, $Q_{n+1} \subset Q_n$. If for any $q \geq 1$, $v \in L^q(\Omega_T)$ and there exist some constants $\alpha_0 \geq 0, \lambda, C_0, C_1 > 0, K > 1$, such that the following inequality holds,*

$$\int\int_{Q_{n+1}} |v|^{\alpha_0 + \lambda K^{n+1}} dxdt \leq \left(C_0 C_1^n \int\int_{Q_n} |v|^{\alpha_0 + \lambda K^n} dxdt \right)^K.$$

Then

$$\operatorname*{ess\,sup}_{Q_n} |v| \leq \left(C_0^{\frac{K}{K-1}} \bar{C}_1 \int\int_{Q_{n_0}} |v|^{\alpha_0 + \lambda K^{n_0}} dxdt \right)^{\frac{1}{\lambda K^{n_0}}}$$

holds, where

$$\bar{C}_1 = C_1^{K_1},$$

$$K_1 = \sum_{n=n_0}^{\infty} n K^{-(n-n_0)},$$

for a $n_0 \in N^+$.

By the above two lemmas, we are able to get the following theorem.

Theorem 6 *Let Ω be a uniformly C^1 domain. If u is the unique solution of the equation (1) in Q_T, and $p > \max\{1, \frac{2N}{N+1}\}$, then*

$$\frac{\partial u}{\partial x_i} \in L^\infty_{\mathrm{loc}}(Q_T), \quad i = 1, 2, \cdots, N. \tag{7}$$

Proof: Since u is the unique solution of equation (1) in Q_T, then we can assume that u is the limitation of following regulation equation's solutions.

$$\frac{\partial u}{\partial t}=\operatorname{div}[(d+\frac{1}{n})^{\alpha}(|\nabla u|^2+\frac{1}{n})^{\frac{p-2}{2}}\nabla u] \quad (8)$$

$$(x,t)\in Q_T=\Omega\times(0,T),$$

$$u(x,0)=u_{0,n}(x), x\in\Omega, \quad (9)$$

$$u(x,t)=g(x,t),(x,t)\in\partial\Omega\times(0,T), \quad (10)$$

where $u_{0,n}(x)$ is the smoothly mollified functions of $u_0(x)$. Let $K\subset Q_T$ be an any compact set. Similar to the proof Lemmas 2.3 in the Chapter 2 of reference [1], which discusses the equation (2), we are able to get

$$\int_K |\nabla u_n|^q dxdt \le C(q,K,p,N). \quad (11)$$

Denote $B_\rho(x_0)=\{x:|x-x_0|<\rho\}$, and for simplicity, denote u_n as u.

Now, we differentiate (8) about x_j, and obtain

$$\frac{\partial u_{x_j}}{\partial t}=\alpha\operatorname{div}[(d+\frac{1}{n})^{\alpha-1}d_{x_j}(|\nabla u|^2+\frac{1}{n})^{\frac{p-2}{2}}\nabla u]$$
$$+[(d+\frac{1}{n})^{\alpha}(|\nabla u|^2+\frac{1}{n})^{\frac{p-2}{2}}u_{x_i}]_{x_ix_j}. \quad (12)$$

Let

$$Tn=\frac{t_0}{2}-\frac{t_0}{2^{n+2}},$$

$$\rho_n=\rho+\frac{\rho}{2^n},$$

$$\overline{\rho}_n=\frac{1}{2}(\rho_n+\rho_{n+1})=\rho+\frac{3\rho}{2^{n+2}},$$

$$B_n=B_{\rho_n}(x_0),$$

$$B'_n=B_{\overline{\rho_n}}(x_0),$$

$$Q_n=B_n\times(T_n,T),$$

$$Q'_n=B'_n\times(T_{n+1},T).$$

Assume that ξ_n is the cut off functions smoothly in Q_n, $\xi_n(\cdot,t)\in C_0^1(B_n)$, then

$$\xi_n=0,\forall t\le T_n,$$

$$\xi_n=1,\forall(x,t)\in Q_n^{'}.$$

$$|\nabla\xi_n|\le\frac{2^{n+2}}{\rho},$$

$$0\le\xi_{nt}\le\frac{2^{n+3}}{t_0}.$$

Let $x_0\in\Omega$, $B_{4\rho}(x_0)=B_{4\rho}\subset\Omega$. We choose $v=|\nabla u|^2+\frac{1}{n}$, multiply the two sides of (12) with $\xi_n^2 v^\beta u_{x_j}$, and integrate on $B_{2\rho}\times(\frac{t_0}{4},t)$, then we can obtain the following equalities

$$\begin{aligned}
&\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_n^2 v^\beta u_{x_j}\frac{\partial u_{x_j}^2}{\partial t}dtdx\\
=&\frac{1}{2(\beta+1)}\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_n^2\frac{\partial v^{\beta+1}}{\partial t}dtdx\\
=&\frac{1}{2(\beta+1)}\int_{B_{2\rho}}\xi_n^2 v^{\beta+1}dtdx\\
&-\frac{1}{\beta+1}\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_{nt}\xi_n v^{\beta+1}dtdx, \quad (13)
\end{aligned}$$

$$\begin{aligned}
&\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_n^2 v^\beta u_{x_j}[(d+\frac{1}{n})^\alpha v^{\frac{p-2}{2}}u_{x_i}]_{x_ix_j}dxdt\\
=&-\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_n^2[(d+\frac{1}{n})^\alpha v^{\frac{p-2}{2}}u_{x_i}]_{x_j}[v^\beta u_{x_j}]_{x_i}dxdt\\
-&\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_n^2(d+\frac{1}{n})^\alpha(v^{\frac{p-2}{2}}u_{x_i})_{x_j}(v^\beta u_{x_j})_{x_i}dxdt\\
-&2\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_{nx_i}\xi_n v^\beta u_{x_j}[(d+\frac{1}{n})^\alpha v^{\frac{p-2}{2}}u_{x_i}]_{x_j}dxdt.\\
-&\alpha\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_n^2 d_{x_j}(d+\frac{1}{n})^{\alpha-1}v^{\frac{p-2}{2}}u_{x_i}(v^\beta u_{x_j})_{x_i}dxdt\\
-&2\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_n\xi_{nx_i}(d+\frac{1}{n})^\alpha(v^{\frac{p-2}{2}}u_{x_i})_{x_j}v^\beta u_{x_j}dxdt\\
-&2\alpha\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_n\xi_{nx_i}d_{x_j}(d+\frac{1}{n})^{\alpha-1}v^{\frac{p-2}{2}}u_{x_i}v^\beta u_{x_j}dxdt \quad (14)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2(\beta+1)}\int_{B_{2\rho}}\xi_n^2 v^{\beta+1}dxdt\\
+&\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_n^2(d+\frac{1}{n})^\alpha(v^{\frac{p-2}{2}}u_{x_i})_{x_j}(v^\beta u_{x_j})_{x_i}dxdt\\
=&-\alpha\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_n^2 d_{x_j}(d+\frac{1}{n})^{\alpha-1}v^{\frac{p-2}{2}}u_{x_i}(v^\beta u_{x_j})_{x_i}dxdt\\
-&2\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_n\xi_{nx_i}(d+\frac{1}{n})^\alpha(v^{\frac{p-2}{2}}u_{x_i})_{x_j}v^\beta u_{x_j}dxdt\\
-&2\alpha\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_n\xi_{nx_i}d_{x_j}(d+\frac{1}{n})^{\alpha-1}v^{\frac{p-2}{2}}u_{x_i}v^\beta u_{x_j}dxdt\\
+&\alpha\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_n^2 v^\beta u_{x_j}[(d+\frac{1}{n})^{\alpha-1}d_{x_j}v^{\frac{p-2}{2}}u_{x_i}]_{x_i}dxdt\\
+&\frac{1}{\beta+1}\int_{\frac{t_0}{4}}^t\xi_n\xi_{nt}v^{\beta+1}dxdt. \quad (15)
\end{aligned}$$

Using the fact that

$$|d_{x_j}|\le|\nabla d|=1,\forall x\in\Omega. \quad (16)$$

and by Young inequality, we have

$$\begin{aligned}
&v^{\frac{p-2}{2}}u_{x_i}v^{\beta-1}u_{x_j}\\
&\le\varepsilon v^{\frac{p+2\beta-4}{2}}|\nabla u|^2+c(\varepsilon)v^{\frac{p+2\beta}{2}},
\end{aligned}$$

$$v^{\frac{p-2}{2}}u_{x_i}v^{\beta}u_{x_ix_j} \le \varepsilon v^{\frac{p+2\beta-2}{2}}u_{x_ix_j}u_{x_ix_j}+c(\varepsilon)v^{\frac{p+2\beta}{2}},$$

$$\xi_{nx_i}v^{\beta}u_{x_j}(v^{\frac{p-2}{2}}u_{x_i})_{x_j} \le \varepsilon|\nabla\xi_n|^2v^{\frac{p+2\beta}{2}} +\varepsilon v^{\frac{p+2\beta-4}{2}}|\nabla v|^2+\varepsilon v^{\frac{p+2\beta-2}{2}}u_{x_ix_j}u_{x_ix_j}.$$

and

$$(v^{\frac{p-2}{2}}u_{x_i})_{x_j}(v^{\beta}u_{x_j})_{x_i}=v^{\frac{p+2\beta-2}{2}}u_{x_ix_j}u_{x_ix_j} +\frac{p+2\beta-2}{4}v^{\frac{p+2\beta-4}{2}}|\nabla v|^2 +\frac{\beta(p-2)}{2}v^{\frac{p+2\beta-6}{2}}(\nabla u\cdot\nabla v)^2.$$

From these formulas and (15), we can obtain

$$\begin{aligned}&\frac{1}{2(\beta+1)}\int_{B_{2\rho}}\xi_n^2v^{\beta+1}dx\\ &+(1-c(\varepsilon))\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_n^2v^{\frac{p+2\beta-2}{2}}u_{x_ix_j}u_{x_ix_j}dxdt\\ &+(\frac{p+2\beta-2}{4}-c(\varepsilon))\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_n^2v^{\frac{p+2\beta-4}{2}}|\nabla v|^2dxdt\\ &+\frac{\beta(p-2)}{2}\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_n^2v^{\frac{p+2\beta-6}{2}}(\nabla u\cdot\nabla v)^2dxdt\\ &\le c(\varepsilon)\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}v^{\frac{p+2\beta}{2}}(1+|\nabla\xi_n|^2)dxdt\\ &+\frac{c}{\beta+1}\int_{\frac{t_0}{4}}^t\int_{B_{2\rho}}\xi_nv^{\beta+1}dxdt. \qquad (17)\end{aligned}$$

where ε is a appropriately small positive constant.

Case 1). When $p\ge 2$, denote

$$w=v^{\frac{p+2\beta}{2}},\lambda=\frac{4(\beta+1)}{p+2\beta}.$$

From (17), we can obtain

$$\begin{aligned}&\sup_{T_n<t<T}\int_{B_n}(\xi_n^{\frac{2}{\lambda}}w)^{\lambda}dx+\int\int_{Q_n}(\xi_n^{\frac{2}{\lambda}}|\nabla w|)^2dxdt\\ &\le C\left[(1+\frac{2^{2n}}{\rho^2})\int\int_{Q_n}w^2dxdt+\frac{2^n}{t_0}\int\int_{Q_n}w^{\lambda}dxdt\right] \qquad (18)\end{aligned}$$

by Lemma 4 and (18),

$$\begin{aligned}&\int\int_{Q_{n+1}}w^{2+\frac{2\lambda}{N}}dxd\tau\\ \le\ & C\left[(1+\frac{2^{2n}}{\rho^2})\int\int_{Q_n}w^2dxdt+\frac{2^n}{t_0}\int\int_{Q_n}w^{\lambda}dxdt\right]^k,\end{aligned}$$

which implies that

$$\begin{aligned}&\int\int_{Q_{n+1}}v^{\frac{p+2\beta}{2}+\frac{2\beta+2}{N}}dxd\tau\\ \le\ & C\left[(1+\frac{2^{2n}}{\rho^2})\int\int_{Q_n}v^{\frac{p+2\beta}{2}}dxd\tau +\frac{2^n}{t_0}\int\int_{Q_n}v^{\beta+1}dxd\tau\right]^{1+\frac{2}{N}},\end{aligned}$$

where $k=1+\frac{2}{N}$. By choosing $2\beta=k^n-2$, the above formula can be changed into

$$\begin{aligned}&\int\int_{Q_{n+1}}v^{\frac{p-2}{2}+\frac{k^{n+1}}{2}}dxd\tau\\ \le\ & C\left[(1+\frac{2^{2n}}{\rho^2})\int\int_{Q_n}v^{\frac{p-2}{2}+\frac{k^n}{2}}dxd\tau +\frac{2^n}{t_0}\int\int_{Q_n}v^{\frac{k^n}{2}}dxd\tau\right]^k. \qquad (19)\end{aligned}$$

If

$$\frac{2^n}{t_0}\int\int_{Q_n}v^{\frac{k^n}{2}}dxd\tau \ge (1+\frac{2^{2n}}{\rho^2})\int\int_{Q_n}v^{\frac{p-2}{2}+\frac{k^n}{2}}dxd\tau, \qquad (20)$$

then by Hölder inequality, we see that

$$\int\int_{Q_n}v^{\frac{p-2}{2}+\frac{k^n}{2}}dxd\tau\le(\frac{\rho^2}{t_0})^{\frac{p-2+k^n}{p-2}}\mathrm{mes}Q_n,$$

which implies that

$$\sup_{B_\rho\times(\frac{t_0}{2},T)}v\le(\frac{\rho^2}{t_0})^{\frac{2}{p-2}}. \qquad (21)$$

Then we can obtain Theorem 6.

If (20) isn't true, we have

$$\begin{aligned}&\int\int_{Q_{n+1}}v^{\frac{p-2}{2}+\frac{k^{n+1}}{2}}dxd\tau\\ &\le C\left[(1+\frac{2^{2n}}{\rho^2})\int\int_{Q_n}v^{\frac{p-2}{2}+\frac{k^n}{2}}dxd\tau\right]^k.\end{aligned}$$

By Lemma 5,

$$\sup_{Q_n}v\le C\rho^{-(N+2)}\int\int_{Q_{n_0}}v^{\frac{p-2}{2}+\frac{k^{n_0}}{2}}dxd\tau, \qquad (22)$$

where n_0 is a positive integer which makes $k^{n_0}>2$ hold. Then we can obtain Theorem 6 according to (11). Therefore, when $p>2$,

$$\frac{\partial u}{\partial u_i}\in L^\infty_{\mathrm{loc}}(Q_T).$$

Case 2). When $p < 2$, we can get

$$v^{\frac{p+2\beta-2}{2}} \sum_{j=1}^{N} |\nabla u_{x_j}|^2 \geq \frac{1}{4} v^{\frac{p+2\beta-4}{2}} |\nabla v|^2$$

and

$$\frac{\beta(p-2)}{2} \int_0^{t_0} \int_{B_{2\rho}} \xi_n^2 v^{\frac{p+2\beta-6}{2}} (\nabla u \cdot \nabla v)^2 dx d\tau \geq \frac{\beta(p-2)}{2} \int_0^{t_0} \int_{B_{2\rho}} \xi_n^2 v^{\frac{p+2\beta-4}{2}} |\nabla u|^2 dx d\tau$$

Then according to (17),

$$\begin{aligned} &\sup_{\frac{t_0}{4}<t<T} \int_{B_{2\rho}} \xi_n^2 v^{\beta+1}(x,\tau) dx \\ &+ \int\int_{Q(2\rho,\frac{t_0}{4})} \xi_n^2 v^{\alpha_p} |\nabla v|^2 dx d\tau \\ &\leq C \int\int_{Q(2\rho,\frac{t_0}{4})} v^{\alpha_p+2} |\nabla \xi_n|^2 dx d\tau \\ &+ C \int\int_{Q(2\rho,\frac{t_0}{4})} \xi_n \xi_{nt} v^{\beta+1} dx d\tau, \end{aligned}$$

where $\alpha_p = \frac{p+2\alpha-4}{2}$. Then using Lemma 4, similar to the discussion of the case (1), we can obtain

$$\begin{aligned} &\left[\int\int_{Q_{n+1}} v^{\frac{N(2-p)}{4}+\frac{k^{n+1}}{2}} dx d\tau\right]^{\frac{1}{k}} \\ \leq\ & C\frac{2^{2n}}{t_0} \int\int_{Q_n} v^{\frac{N(2-p)}{4}+\frac{k^n}{2}} dx d\tau \\ +\ & C(1+\frac{2^{2n}}{\rho^2}) \int\int_{Q_n} v^{\frac{(N-2)(2-p)}{4}+\frac{k^n}{2}} dx d\tau. \end{aligned} \quad (23)$$

If

$$\begin{aligned} &(1+\frac{2^{2n}}{\rho^2}) \int\int_{Q_n} v^{\frac{(N-2)(2-p)}{4}+\frac{k^n}{2}} dx d\tau \\ \geq\ & \frac{2^{2n}}{t_0} \int\int_{Q_n} v^{\frac{N(2-p)}{4}+\frac{k^n}{2}} dx d\tau, \end{aligned} \quad (24)$$

then by Hölder inequality, we have

$$\begin{aligned} &\int\int_{Q_n} v^{\frac{N(2-p)}{4}+\frac{k^n}{2}} dx d\tau \\ &\leq (\frac{t_0+\rho^2}{\rho^2})^{\frac{N(2-p)+k^n}{(2-p)}} \mathrm{mes} Q_n, \end{aligned}$$

which implies that

$$\sup_{B_\rho \times (\frac{t_0}{2},T)} v \leq (\frac{t_0}{\rho^2})^{\frac{2}{2-p}}.$$

If (24) isn't true, then from (23), we have

$$\begin{aligned} &\left[\int\int_{Q_{n+1}} v^{\frac{N(2-p)}{4}+\frac{k^{n+1}}{2}} dx d\tau\right]^{\frac{1}{k}} \\ &\leq \gamma(1+\frac{2^{2n}}{\rho^2}) \int\int_{Q_n} v^{N(2-p)+\frac{k^n}{2}} dx d\tau. \end{aligned}$$

Then using Lemma 5, we obtain

$$\begin{aligned} &\sup_{B_\rho \times (\frac{t_0}{2},T)} v \\ &\leq C\left[\rho^{-(N+2)} \int\int_{Q_{n_0}} v^{\frac{N(2-p)}{4}+\frac{k^{n_0}}{2}} dx d\tau\right]^{\frac{2}{k^{n_0}}}. \end{aligned}$$

Also we can obtain Theorem 6 according to (11).

Therefore, when $p < 2$, also

$$\frac{\partial u}{\partial u_i} \in L^\infty_{\mathrm{loc}}(Q_T).$$

The proof of Theorem 6 is complete.

3 The continuity of the solution

Theorem 7 *Supposed that u is a weak solution of equation (1) in Q_T, then for any compact set $K \subset Q_T$, $(x_1,t_1), (x_2,t_2) \in K$,*

$$\begin{aligned} &|u(x_1,t_1) - u(x_2,t_2)| \\ &\leq c(|x_1-x_2| + |t_1-t_2|^{\frac{1}{2}}), \end{aligned} \quad (25)$$

where c is a constant only dependent on N, p, $d(K,\partial\Omega)$ and $\|u\|_{L^\infty(K)}$.

Proof: Obviously we only need to prove that u satisfies (25) in $B_R \times (t_0,T)$, for any $R > 0$ such that $B_R \subset \Omega$, $t_0 \in (0,T)$. Let u_ε be the usual mollified function of u,

$$\begin{aligned} u_\varepsilon(x,t) &= J_\varepsilon * u(x,t) \\ &= \int_0^T \int_{R^N} j_\varepsilon(x-y,t-\tau) u(y,\tau) dy d\tau, \end{aligned}$$

where $0 < \varepsilon < t_0 < t < T-\varepsilon$. Then for any $x_1, x_2 \in B_R$, we obtain

1).
$$\begin{aligned} &u_\varepsilon(x_1,t) - u_\varepsilon(x_2,t) \\ &= \int_0^T \int_{R^N} j_\varepsilon(x_1-y,t-\tau) u(y,\tau) dy d\tau \\ &- \int_0^T \int_{R^N} j_\varepsilon(x_2-y,t-\tau) u(y,\tau) dy d\tau \\ &= \int_0^T \int_{R^N} [j_\varepsilon(x_1-y,t-\tau) - j_\varepsilon(x_2-y,t-\tau)] \\ &\qquad \cdot u(y,\tau) dy d\tau \end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \int_{R^N} \int_0^1 \frac{d[j_\varepsilon(sx_1+(1-s)x_2-y,t-\tau)]}{ds} \\
&\qquad \cdot u(y,\tau)dsdyd\tau \\
&= \int_0^T \int_{R^N} \int_0^1 \nabla_x[j_\varepsilon(sx_1+(1-s)x_2-y,t-\tau)] \\
&\qquad \cdot u(y,\tau)dsdyd\tau(x_1-x_2) \\
&= -\int_0^T \int_{R^N} \int_0^1 \nabla_y[j_\varepsilon(sx_1+(1-s)x_2-y,t-\tau)] \\
&\qquad \cdot u(y,\tau)dsdyd\tau(x_1-x_2) \\
&= \int_0^T \int_{R^N} \int_0^1 j_\varepsilon(sx_1+(1-s)x_2-y,t-\tau) \\
&\qquad \nabla_y u(y,\tau)dsdyd\tau(x_1-x_2). \qquad (26)
\end{aligned}
$$

According to Theorem 6, we have

$$
\begin{aligned}
& |u_\varepsilon(x_1,t)-u_\varepsilon(x_2,t)| \\
& \leq \int_0^T \int_{R^N} \int_0^1 |j_\varepsilon(sx_1+(1-s)x_2-y,t-\tau)| \\
&\qquad |\nabla_y u(y,\tau)|dsdyd\tau|x_1-x_2| \\
\leq\ & c|x_1-x_2|, \qquad (27)
\end{aligned}
$$

where and in what follows, c is a constant independent of ε.

2). Let $0<\varepsilon<t_0<t_1<t_2<T$, $B(\triangle t)=B_{(\triangle t)^{\frac{1}{2}}}(x_0)$, $\varphi\in C_0^1(B(\triangle t))$, $x_0\in B_R$, $\triangle t=t_2-t_1$. Then

$$
\begin{aligned}
& \int_{B(\triangle t)} \varphi(x)[u_\varepsilon(x,t_2)-u_\varepsilon(x,t_1)]dx \\
&= \int_{B(\triangle t)} \varphi(x) \int_0^1 \frac{du_\varepsilon(x,st_2+(1-s)t_1)}{ds} dsdx \\
&= \triangle t \int_{B(\triangle t)} \varphi(x) \int_0^1 \int_0^T \int_{R^N} \\
& j_{\varepsilon t}(x-y,st_2+(1-s)t_1-\tau)u(y,\tau)dyd\tau dsdx \\
&= -\triangle t \int_{B(\triangle t)} \varphi(x) \int_0^1 \int_0^T \int_{R^N} \\
& j_{\varepsilon\tau}(x-y,st_2+(1-s)t_1-\tau)u(y,\tau)dyd\tau dsdx.
\end{aligned}
$$

For fixed $(x,t)\in Q_t$, $0<\varepsilon<t_0<t<T-\varepsilon$, $J_\varepsilon(x-y,t-\tau)\in C_0^1(Q_T)$. Let us choose the test function in the definition of generalized solution (6) as $\varphi(x)=J_\varepsilon(x-y,t-\tau)$

$$
\int_{Q_T} (\frac{\partial u}{\partial t}\varphi + d^\alpha\cdot|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi)dxdt = 0.
$$

Then, we have

$$
\begin{aligned}
& \int_0^T \int_{R^N} j_{\varepsilon\tau}(x-y,st_2+(1-s)t_1-\tau)u(y,\tau)dyd\tau \\
&= \int_0^T \int_{R^N} d^\alpha\cdot|\nabla_y u|^{p-2}\nabla_y u \\
&\qquad \nabla_y j_\varepsilon(x-y,st_2+(1-s)t_1-\tau)dyd\tau.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
& \int_{B(\triangle t)} \varphi(x)[u_\varepsilon(x,t_2)-u_\varepsilon(x,t_1)]dx \\
&= -\triangle t \int_{B(\triangle t)} \varphi(x) \int_0^1 \int_0^T \int_{R^N} d^\alpha\cdot|\nabla_y u|^{p-2}\nabla_y u \\
&\quad \nabla_y j_\varepsilon(x-y,st_2+(1-s)t_1-\tau)dyd\tau dsdx \\
&= -\triangle t \int_0^1 \int_0^T \int_{R^N} d^\alpha\cdot|\nabla_y u|^{p-2}\nabla_y u \int_{B(\triangle t)} \\
& \nabla_x\varphi j_\varepsilon(x-y,st_2+(1-s)t_1-\tau)dyd\tau dsdx \\
&= -\triangle t \int_0^1 \int_{B(\triangle t)} \nabla_x\varphi \int_0^T \int_{R^N} j_\varepsilon(x-y, \\
& st_2+(1-s)t_1-\tau)d^\alpha\cdot|\nabla_y u|^{p-2}\nabla_y u dyd\tau dxds \\
&= -\triangle t \int_0^1 \int_{B(\triangle t)} \nabla_x\varphi J_\varepsilon(d^\alpha\cdot|\nabla_y u|^{p-2}\nabla_y u) \\
&\qquad \cdot(x,st_2+(1-s)t_1)dxds. \qquad (28)
\end{aligned}
$$

Let $\delta(s)\in C_0^1(R)$ satisfy

$$
\delta(s)\geq 0, \quad \int_R \delta(s)ds = 1
$$

and $\delta(s)=0$ when $s\geq 1$. For any $h>0$, we define

$$
\delta_h(s) = \frac{\delta(\frac{s}{h})}{h}.
$$

By a approaching process, we know that (28) is also true for any $\varphi\in W_0^{1,1}(B(\triangle t))$. Choosing

$$
\varphi = \varphi_h(x) = \int_{-h}^{(\triangle t)^{\frac{1}{2}}-|x-x_0|-2h} \delta_h(s)ds
$$

in (28), then

$$
\begin{aligned}
& \int_{B(\triangle t)} \varphi_h(x)(u_\varepsilon(x,t_2)-u_\varepsilon(x,t_1))dx \\
&= -\triangle t \int_{B(\triangle t)} \delta_h((\triangle t)^{\frac{1}{2}}-|x-x_0|-2h)\cdot\frac{x_{0i}-x_i}{|x-x_0|} \\
&\quad \cdot J_\varepsilon(d^\alpha\cdot|\nabla_y u|^{p-2}\nabla_y u)(x,st_2+(1-s)t_1)dxds. \qquad (29)
\end{aligned}
$$

We notice that, when $x\in B(\triangle t)$,

$$
\lim_{h\to 0}\varphi_h(x) = 0.
$$

But

$$
\delta_h((\triangle t)^{\frac{1}{2}}-|x-x_0|-2h) = 0
$$

holds when $|x-x_0|<(\triangle t)^{\frac{1}{2}}-h$, then $\delta_h\leq\frac{c}{h}$ and

$$
\mathrm{mes}(B(\triangle t)\backslash B_{(\triangle t)^{\frac{1}{2}}-h}(x_0)) \leq ch(\triangle t)^{\frac{N-1}{2}}.
$$

Therefore using Theorem 6 and (29), we can obtain

$$\left|\int_{B(\triangle t)} \varphi_h(x)[u_\varepsilon(x,t_2)-u_\varepsilon(x,t_1)]dx\right| \le c(\triangle t)^{\frac{N+1}{2}}.$$

Letting $h\to 0$, we obtain

$$\left|\int_{B(\triangle t)} [u_\varepsilon(x,t_2)-u_\varepsilon(x,t_1)]dx\right| \le c(\triangle t)^{\frac{N+1}{2}}.$$

Then by the mean value theorem, there exist $x^*\in B(\triangle t)$ such that

$$|u_\varepsilon(x^*,t_2)-u_\varepsilon(x^*,t_1)| \le c(\triangle t)^{\frac{1}{2}}.$$

Noticing that

$$\begin{aligned}&|u_\varepsilon(x_0,t_2)-u_\varepsilon(x_0,t_1)|\\ &\le |u_\varepsilon(x_0,t_2)-u_\varepsilon(x^*,t_2)|+|u_\varepsilon(x^*,t_2)\\ &\qquad -u_\varepsilon(x^*,t_1)|+|u_\varepsilon(x^*,t_1)-u_\varepsilon(x_0,t_1)|\\ &\le c(\triangle t)^{\frac{1}{2}},\end{aligned}$$

combining this formula with (26), letting $\varepsilon\to 0$, we obtain the following formula

$$\begin{aligned}&|u(x_1,t_1)-u(x_2,t_2)|\\ &\le |u(x_1,t_1)-u(x_2,t_1)|+|u(x_2,t_1)-u(x_2,t_2)|\\ &\le c(|x_1-x_2|+|t_1-t_2|^{\frac{1}{2}}).\end{aligned}$$

The proof is complete.

4 Continuity of the gradient of the solution

In what follows, let u be the solution of (1) and let $P_0=(x_0,t_0)\in Q_T, 0<R\le 1, \mu>1$. If we denote

$$\begin{aligned}&Q_\mu(P_0,R)\\ &=\left\{(x,t):|x-x_0|<R, t_0-\frac{R^2}{\mu^{p-2}}<t<t_0\right\},\end{aligned}$$

$$M_{i\mu}^{\pm}(R)=\operatorname{ess}\sup_{Q_\mu(P_0,R)}(\pm u_{x_i}), i=1,2,\ldots,N.$$

$$M_\mu(R)=\max_{1\le i\le N}\operatorname{ess}\sup_{Q_\mu(P_0,R)}|u_{x_i}|,$$

then we can get the following propositions similarly as Chapter 2 in [1], we omit the details here.

Proposition 8 *Assume that*

$$2M_{1\mu}^{+}(R)\ge M_\mu(R)$$

and μ satisfies

$$2M_{1\mu}^{+}(R)\ge \mu\ge M_\mu(R).$$

Then there exists $\varepsilon_0=\varepsilon_0(p,N)$ such that, when

$$\begin{aligned}\frac{1}{\mathrm{mes}Q_u(P_0,R)}\int\int Q_\mu(P_0,R)(M_{1\mu}^{+}(R)-u_{x_1})^2dxdt\\ \le \varepsilon_0(M_{1\mu}^{+}(R)^2,\end{aligned}$$

it holds that

$$\operatorname{ess}\sup_{Q_\mu(P_0,\frac{R}{2})} u_{x_1}\ge \frac{M_{1\mu}^{+}(R_2)}{2}.$$

Proposition 9 *Assume that*

$$2M_{1\mu}^{+}(R)\ge M_\mu(R)$$

and μ satisfies

$$2M_{1\mu}^{+}(R)\ge \mu\ge M_\mu(R).$$

Then for any $\varepsilon_0>0$, there always exist some constants $\lambda,\beta\in(0,1)$ which depend on p,N,ε_0 such that, if

$$\begin{aligned}\frac{1}{\mathrm{mes}Q_u(P_0,R)}\int\int Q_\mu(P_0,R)(M_{1\mu}^{+}(R)-u_{x_1})^2dxdt\\ >\varepsilon_0(M_{1\mu}^{+})^2,\end{aligned}$$

then

$$\begin{aligned}\mathrm{mes}\left\{(x,t)\in Q_\mu(P_0,R):u_{x_1}(x,t)\le(1-\beta)M_{1\mu}^{+}(R)\right\}\\ >\lambda\mathrm{mes}Q_\mu(P_0,R).\end{aligned}$$

Proposition 10 *Assume that*

$$2M_{1\mu}^{+}(R)\ge M_\mu(R)$$

and μ satisfies

$$2M_{1\mu}^{+}(R)\ge \mu\ge M_\mu(R).$$

Further assume that there exist some constants $\lambda,\beta\in(0,1)$ such that, if

$$\begin{aligned}\mathrm{mes}\left\{(x,t)\in Q_\mu(P_0,R):u_{x_1}(x,t)\le(1-\beta)M_{1\mu}^{+}(R)\right\}\\ \ge\lambda\mathrm{mes}Q_\mu(P_0,R),\end{aligned}$$

then there exist constants $\delta,\gamma\in(0,1)$ which depend on p,N,λ,β such that

$$M_{1\mu}^{+}(\delta R)\le\gamma M_{1\mu}^{+}(R).$$

Theorem 11 *Let $p > \max\{1, \frac{2N}{N+2}\}, u$ is the generalized solution of equation (1) in Q_T, then $u_{x_j}(j = 1, 2, \cdots, N)$ is locally Hölder continuous in Q_T.*

Proof: The proof of the theorem includes three steps. Firstly, choose the ε_0 on Proposition 8. Secondly, determine λ, β on Proposition 9 according to ε_0. Lastly, determine δ, γ on Proposition 10 according to λ, β. In particular, δ, γ is just dependent on N, p.

Choosing $s \in (1, 2)$, which is close to 2, such that

$$\delta^{\frac{2(2-s)}{s(p-2)}} > \max\{\frac{1}{2}, \gamma\}. \quad (30)$$

Letting $0 < \eta_0 < T$, and $\Omega_{\eta_0 T} = \Omega \times (\eta_0, T), \Omega_{\eta_0 T} \subset\subset Q_T$ be a bounded open set. Then we obtain that ∇u is bounded in $\Omega_{\eta_0 T}$ according to Theorem 6. We may assume

$$\|\nabla u\|_{\infty\ \Omega_{\eta_0 T}} \leq \overline{M_0}. \quad (31)$$

Denote $M = \overline{M_0}\delta^{\frac{-2(2-s)}{s(p-2)}}$, then $\overline{M_0} = M\delta^{\frac{2(2-s)}{s(p-2)}}$. Choose $R_0 \in (0, 1]$ such that

$$Q_{2M_0}(P_0, R_0) \subset \Omega_{\eta_0 T}.$$

For any $0 < R \leq R_0$, denote

$$t_R = R^s R_0^{2-s}(2M_0)^{2-p},$$

$$\hat{Q}(P_0, R) = \{(x, t); |x - x_0| < R, t_0 - t_R < t < t_0\},$$

$$M_i^{\pm}(R) = \operatorname{ess}\sup_{\hat{Q}(P_0,R)} (\pm u_{x_i}), \quad i = 1, 2, \cdots, N;$$

$$M(R) = \max_{1\leq i\leq N} \operatorname{ess}\sup_{\hat{Q}(P_0,R)} |u_{x_i}|,$$

$$\begin{aligned}\operatorname{osc}_{\hat{Q}(P_0,R)} u_{x_i} &= \operatorname{ess}\sup_{\hat{Q}(P_0,R)} u_{x_i} - \operatorname{ess}\inf_{\hat{Q}(P_0,R)} u_{x_i} \\ &= M_i^+(R) + M_i^-(R).\end{aligned}$$

In what follows, we will prove that there exist constants $\rho \in (0, 1)$ and $C > 0$ which only depend on N, p, such that

$$\operatorname{osc}_{\hat{Q}(P_0,R)} u_{x_i} \leq CM_0(\frac{R}{R_0})^{\rho}, \quad \forall 0 < R < R_0. \quad (32)$$

By (32), we immediately know that $u_{x_i}(i = 1, 2 \ldots, N)$ is Hölder continue in $\Omega_{\eta_0 T}$, and know that the theorem is true.

Now we prove (32). Define

$$\begin{aligned}R_1 = \ &\sup\{R \in [0, R_0]; \text{exist } 1 \leq j \leq N, \theta \in \{+, -\}, \\ &\text{such that} |M_j^{\theta}(R)| \geq 2M_0(\tfrac{R}{R_0})^{\frac{2-s}{p-2}}\}. \quad (33)\end{aligned}$$

Assume $R_1 > 0$ without loss of the generality. Otherwise,

$$|M_j^{\theta}(R)| < 2M_0(\frac{R}{R_0})^{\frac{2-s}{p-2}}, \quad 1 \leq j \leq N, \theta \in \{+, -\},$$

then we obtain

$$\operatorname{osc}_{\hat{Q}(P_0,R)} u_{x_i} = M_i^+(R) + M_i^-(R) < 4M_0(\frac{R}{R_0})^{\rho},$$

where $\rho = \frac{2-s}{p-2}$. Therefore formula (32) holds naturally. Now according to the definition of $M_0, \overline{M_0}$, we can obtain $0 < R_1 \leq \delta^{\frac{2}{s}} R_0 < R_0$ by (33). Therefore there exists R_2 and $\delta^{\frac{2}{s}} R_2 < R_1 < R_2 < R_0$, such that

$$|M_j^{\pm}(R_2)| \leq 2M_0(\frac{R_2}{R_0})^{\frac{2-s}{p-2}} \quad j = 1, 2 \ldots, N. \quad (34)$$

At the same time, there exist i_0, θ, without loss of the generality, we can choose $i_0 = 1, \theta = +$, such that

$$M_1^+(\delta^{\frac{2}{s}} R_2) > 2M_0(\frac{\delta^{\frac{2}{s}} R_2}{R_0})^{\frac{2-s}{p-2}}. \quad (35)$$

Denote

$$\mu = 2M_0(\frac{R_2}{R_0})^{\frac{2-s}{p-2}}. \quad (36)$$

We will prove the following formula,

$$\begin{aligned}&\int\int Q_\mu(P_0, R_2)(M_1^+(R_2) - u_{x_1})^2 dxdt \\ \leq\ &\varepsilon_0(M_1^+(R_2)^2 \text{mes} Q_\mu(P_0, R). \quad (37)\end{aligned}$$

According to the definition of μ, we clearly have

$$Q_\mu(P_0, R_2) = \hat{Q}(P_0, R_2), \quad M_{1\mu}^+(R_2) = M_1^+(R_2),$$

therefore we can obtain following formula from (35), (36) and (30)

$$M_{1\mu}^+(R_2) \geq M_1^+(\delta^{\frac{2}{s}} R_2) > \max\{\gamma, \frac{1}{2}\}\mu. \quad (38)$$

Then according to (34) and (36), we obtain

$$2M_{1\mu}^+(R_2) > \mu \geq M_\mu(R_2). \quad (39)$$

Let us prove (37) now. If (37) isn't true, then according to Proposition 9 and Proposition 10, we obtain

$$M_{1\mu}^+(\delta R_2) \leq \gamma M_{1\mu}^+(R_2).$$

Noticing that $Q_\mu(P_0, \delta R_2) = \hat{Q}(P_0, \delta^{\frac{2}{s}} R_2)$, then we can obtain

$$M_1^+(\delta^{\frac{2}{s}} R_2) = M_{1\mu}^+(\delta R_2) \leq \gamma M_{1\mu}^+(R_2) \leq \gamma\mu.$$

But this conflicts (38), therefore formula (37) holds.

Now, according to (38) and Proposition 8, we have

$$\operatorname*{ess\ sup}_{Q_\mu(P_0,\frac{R_2}{2})} u_{x_1} \geq \frac{M^+_{1\mu}(R_2)}{2} \geq \frac{\mu}{4}. \tag{40}$$

As follows, according to (40), we will prove that u_{x_m} satisfies (32).

Differentiate (1) about x_m, obtain

$$\frac{\partial u_{x_m}}{\partial t} = \alpha \operatorname{div}(d^{\alpha-1} d_{x_m} |\nabla u|^{p-2}\nabla u) + d^\alpha \cdot (|\nabla u|^{p-2} u_{x_i})_{x_i x_m}.$$

The above formula can be changed into

$$\frac{d^{-\alpha}\partial u_{x_m}}{\partial t} - (a_{ij}|\nabla u|^{p-2} u_{x_m x_j})_{x_i} = \alpha d^{-\alpha}\operatorname{div}(d^{\alpha-1} d_{x_m}|\nabla u|^{p-2}\nabla u), \tag{41}$$

where

$$a_{ij} = \delta_{ij} + \frac{(p-2)u_{x_i}u_{x_j}}{|\nabla u|^2},$$

and δ_{ij} is the Kronecker sign as usual. Let $\xi = x - x_0, \tau = d^\alpha \mu^{p-2}(t-t^0), v(\xi,\tau) = u_{x_m}(x,t)$, $Q'(R) = \{(\xi,\tau); |\xi| < R, -R^2 < \tau \leq 0\}$. Then v satisfies

$$\frac{\partial v}{\partial \tau} - [a_{ij}\frac{|\nabla u|^{p-2}}{\mu^{p-2}} v_{\xi_j}]_{\xi_i} = \alpha d^{-\alpha}\operatorname{div}(d^{\alpha-1} d_{x_m}|\nabla u|^{p-2}\nabla u). \tag{42}$$

Therefore according to (31) and (40), we obtain

$$\frac{1}{C}|\eta|^2 \leq a_{ij}(\frac{|\nabla u|}{\mu})^{p-2}\eta_i\eta_j \leq C|\eta|^2, \quad \forall \eta \in R^N, (\xi,\tau) \in Q'(\frac{R_2}{2}).$$

Since Ω is bounded domain with appropriately smooth boundary, $d(x) = dist(x,\partial\Omega)$ is bounded. Then according to (11) (16) and Theorem 6, we obtain $|\alpha d^{-\alpha}\operatorname{div}(d^{\alpha-1}d_{x_m}|\nabla u|^{p-2}\nabla u)|$ is bounded. This can explain that the formula (42) is uniformly parabolic in $Q'(\frac{R_2}{2})$, then according to reference [5] we obtain

$$\operatorname{osc}_{Q'(R)} v \leq C(\frac{R}{R_2})\operatorname{osc}_{Q'(\frac{R_2}{4})} v, \ \forall \quad 0 < R < \tfrac{R_2}{4},$$

where $C > 0$, $\bar{\beta} \in (0,1)$ is just dependent on N, p.

Backing to the variables (x,t), we obtain

$$\operatorname{osc}_{Q_u(p_0,R)} u_{x_m} \leq C(\frac{R}{R_2})^{\bar\beta}\operatorname{osc}_{Q_u(p_0,\frac{R_2}{4})} u_{x_m} u_{x_m}, \quad \forall 0 < R < \frac{R_2}{4}, \quad m = 1,2\ldots N. \tag{43}$$

1). If $R \geq R_2$, according to the definition of R_2, we obtain

$$\operatorname{osc}_{Q_u(p_0,R)} u_{x_m} \leq |M^+_m(R)| + |M^-_m(R)| \leq 4M_0(\frac{R}{R_0})^{\frac{2-s}{p-2}}. \tag{44}$$

2). If $\frac{R_2}{4} \leq R \leq R_2$, then

$$\operatorname{osc}_{\hat{Q}_u(p_0,R)} u_{x_m} \leq \operatorname{osc}_{\hat{Q}_u(p_0,4R)} u_{x_m} \leq 4M_0(\frac{4R}{R_0})^{\frac{2-s}{p-2}}. \tag{45}$$

3). If $0 < R < \frac{R_2}{4}$, then according to (43) and (45), we obtain

$$\operatorname{osc}_{Q_u(p_0,R)} u_{x_m} \leq C(\frac{R}{R_2})^{\bar\beta} 4M_0(\frac{4R_2}{R_0})^{\frac{2-s}{p-2}}.$$

Letting $\rho = \min\{\bar\beta, \frac{2-s}{p-2}\}$, we obtain

$$\operatorname{osc}_{Q_u(p_0,R)} u_{x_m} \leq CM_0(\frac{R}{R_2})^\rho(\frac{R_2}{R_0})^\rho = CM_0(\frac{R}{R_0})^\rho.$$

Since $\hat{Q}(P_0,R) \subset Q_u(p_0,R)$, the formula (32) holds.

According to 1), 2), 3), we obtain Theorem 11 immediately. The proof is complete.

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