Introduction to the Angular Functions in Euclidian 2D-space

CLAUDE ZIAD BAYEH[1, 2]
[1]Faculty of Engineering II, Lebanese University
[2]EGRDI transaction on Mathematics (2004)
LEBANON
Email: claude_bayeh_cbegrdi@hotmail.com

Abstract: **-** The Angular functions are new mathematical functions introduced by the author, they produce rectangular signals, in which period is function of angles and not of time as the previous functions. Similar to trigonometric functions, the angular functions have the same properties as the precedent, but the difference is that a rectangular signal is obtained instead of a sinusoidal signal, and moreover, one can change the width of each positive and negative alternate in the same period. This is not the case of any other trigonometric function. In other hand, one can change the frequency, the amplitude and the width of any period of the signal at any position by using the general form of the angular function. In this paper, an original study is introduced. Thus, the definition of the original part is presented. The angular functions are also defined. These functions are very important in technical subjects. They will be widely used in mathematics and in engineering domains, especially in power electronics, signal theory, propagation of signals and many other topics. Moreover, the Angular functions are the basis of the Elliptical trigonometry and the rectangular trigonometry in which they are new domains introduced in mathematics by the author.

Key-words: **-** Mathematics, geometry**,** trigonometry, pulse width modulation, signal theory, power electronics.

1 Introduction

In mathematics, signal theory, signal processing and in electronics, there exists many definitions that describe a rectangular signal that has two levels [8]. In fact, square wave, rectangular function, pulse wave, pulse width modulation and many other functions are defined [4]. The main goal of these definitions is to modulate rectangular signals that have exceedingly importance in the digital systems and electronics especially when using the Fourier series and transformation [16], [21]. The angular functions have many advantages ahead the previous functions. Firstly, the expressions are reduced and simplified [1], [2]. Secondly, the angular functions depend on the angle as the trigonometric functions; this is not the case of any other functions in which they depend on the time [3], [4]. Thirdly, the angular functions are controlled functions, in which one can control the width of any period, the width of the positive and negative alternate parts in the same period, the amplitude and the frequency of each period. On other hand, the importance of the angular functions is to unify many expressions that depend on the angle as the case of the elliptical trigonometry [1], [2], [4], [5], and the case of the Rectangular trigonometry [3] which are introduced by the author of this paper. The angular functions are easily programmed and simulated using software as Mat lab, Semolina and Lab view [4], [5].

In this paper, the definitions of the angular functions are presented in the second section. In the third section, the derivative form is presented. In the fourth section, a survey on the InfoMath function is presented in order to facilitate the expressions written using the derivative form of the angular functions. A definition of the step-rectangular waveform "n" order is presented in the section 5. A survey on the applications of the Angular Functions into the electrical engineering domain is presented in the section 6. Finally, a conclusion about the angular functions is presented in section 7.

2 Definitions of the angular functions

Angular functions are denoted using the following abbreviation "$ang_f(\alpha)$":

-the word “ ang ” is related to the angular function name.

-the letter “f” represents the characteristic of the specific angular function (ang_x , ang_y , ang_α , $ang_{|\alpha|}$ and $ang_{\Sigma|\alpha_k|}$). In fact, five principal functions are defined in this paper.

In general, the angular function takes two levels (+1 and -1). The angular function $ang_{\Sigma|\alpha_k|}$, is the most important function of its category, in which one can control the width of the positive and negative part of any period, the amplitude and the frequency.

2.1 Angular function $ang_x(x)$

The expression of the angular function related to the (ox) axis is defined, for $K \in \mathbb{Z}$, as:

$$ang_x(\beta(x+\gamma)) = \begin{cases} +1 \ for \ (4K-1)\frac{\pi}{2\beta}-\gamma \le x \le (4K+1)\frac{\pi}{2\beta}-\gamma \\ -1 \ for \ (4K+1)\frac{\pi}{2\beta}-\gamma < x < (4K+3)\frac{\pi}{2\beta}-\gamma \end{cases} \quad (1)$$

With:

β is the frequency of the function

γ is the translation of the function on the axis (ox).

x is the a variable parameter $x \in]-\infty;+\infty[$

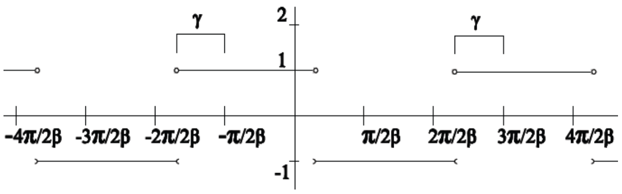


Fig. 1: The $ang_x(\beta(x+\gamma))$ waveform.

In fact:

$$ang_x(x) = \begin{cases} +1 \ for \ (4K-1)\frac{\pi}{2} \le x \le (4K+1)\frac{\pi}{2} \\ -1 \ for \ (4K+1)\frac{\pi}{2} < x < (4K+3)\frac{\pi}{2} \end{cases}$$

For x going from -∞ to +∞ the sign of the function changes into two values +1 and -1 only,

For $x = (2K+1)\frac{\pi}{2}$, it changes from:

$$\begin{cases} -1 \ to +1 \ \ for \ x = -\frac{\pi}{2}+2K\pi = (4K-1)\frac{\pi}{2} \\ +1 \ to -1 \ \ for \ x = +\frac{\pi}{2}+2K\pi = (4K+1)\frac{\pi}{2} \end{cases}$$

• Particular case: for $\beta = 1$ and $\gamma = 0$, the expression (1) becomes:

$$ang_x(x) = \begin{cases} +1 \ for \ \cos(x) \ge 0 \\ -1 \ for \ \cos(x) < 0 \end{cases} \quad (2)$$

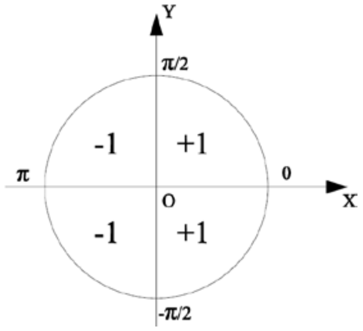


Fig. 2: The $ang_x(x)$ on the unit circle.

2.1.1 Application of the Angular function $ang_x(x)$ in the elliptical trigonometry

The Angular function $ang_x(x)$ has a huge importance in the elliptical trigonometry [4],[5] in which it helps to formulate many equations that can produce huge number of different signals by varying some parameters, the author choose one function as following in order to give an example about the importance of $ang_x(x)$:

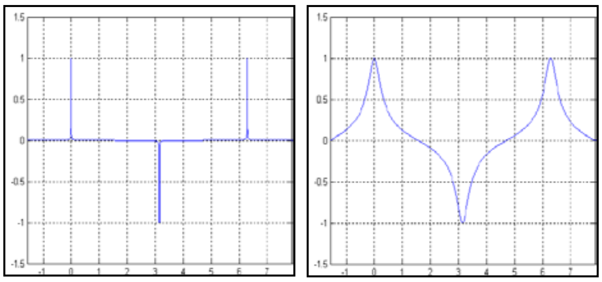
The Absolute Elliptic *Jes* which is defined by the author in the published papers [4],[5] has the following form:

$$\bar{E}jes_{i,b}(x) = \frac{ang_x(x)}{\sqrt{1+\left(\frac{a}{b}\text{Cter}\,(x)\right)^2}} \cdot \left(ang_x(x)\right)^i \quad (3)$$

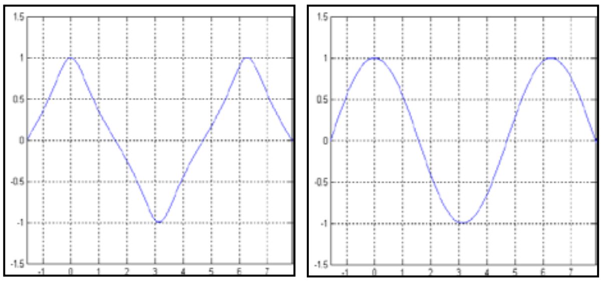
The Absolute Elliptic *Jes* is a powerful function that can produce more than 14 different signals by varying only two parameters i and b. Similar to the cosine function in the traditional trigonometry, the Absolute Elliptical *Jes* is more general than the precedent.

• Multi form signals made by $\bar{E}jes_{i,b}(x)$:

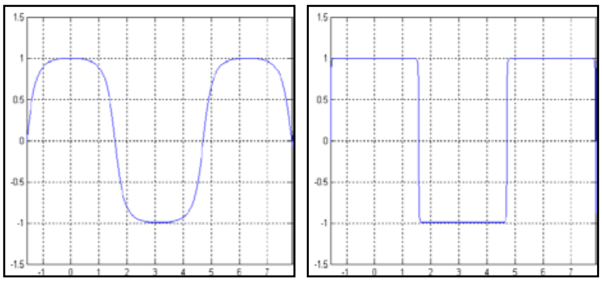
Figures 3 and 4 represent multi form signals obtained by varying two parameters (i and b). For the figures 3.a to 3.f the value of $i = 2$, for the figures 4.a to 4.f the value of $i = 1$.



a) $i = 2, b = 0.001$ b) $i = 2, b = 0.2$

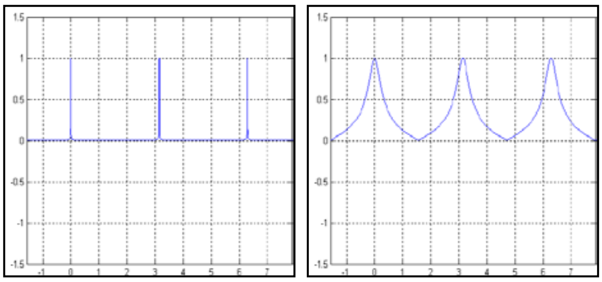


c) $i = 2, b = \sqrt{3}/3$ d) $i = 2, b = 1$

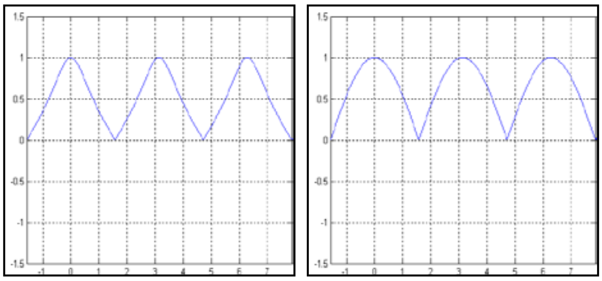


e) $i = 2, b = 3$ f) $i = 2, b = 90$

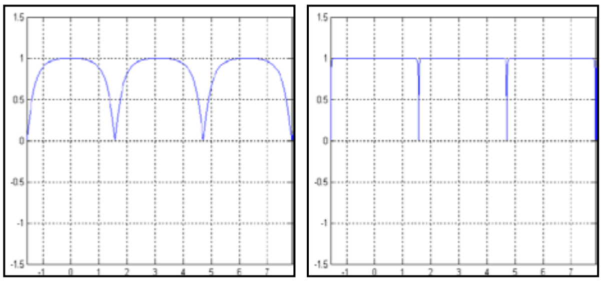
Fig. 3: multi form signals of the function $\bar{E}jes_{i,b}(x)$ for $i = 2$ and for different values of $b > 0$.



a) $i = 1, b = 0.001$ b) $i = 1, b = 0.2$



c) $i = 1, b = \sqrt{3}/3$ d) $i = 1, b = 1$



e) $i = 1, b = 3$ f) $i = 1, b = 90$

Fig. 4: multi form signals of the function $\bar{E}jes_{i,b}(x)$ for $i = 1$ and for different values of $b > 0$.

Important signals obtained using this function:
Impulse train with positive and negative part, elliptic deflated, quasi-triangular, sinusoidal, elliptical swollen, square signal, rectangular signal, impulse train (positive part only), rectified elliptic deflated, saw signal, rectified elliptical swollen, continuous signal…
These types of signals are widely used in power electronics, electrical generator and in transmission of analog signals [21].

2.2 Angular function $ang_y(x)$

The expression of the angular function related to the (oy) axis is defined, for $K \in \mathbb{Z}$, as:

$$ang_y(\beta(x+\gamma)) = \begin{cases} +1 & for\ \ 2K\pi/\beta - \gamma \leq x \leq (2K+1)\pi/\beta - \gamma \\ -1 & for\ \ (2K+1)\pi/\beta - \gamma < x < (2K+2)\pi/\beta - \gamma \end{cases} \quad (4)$$

With:
β is the frequency of the function
γ is the translation of the function on the axis (ox).
x is the a variable parameter $x \in]-\infty; +\infty[$

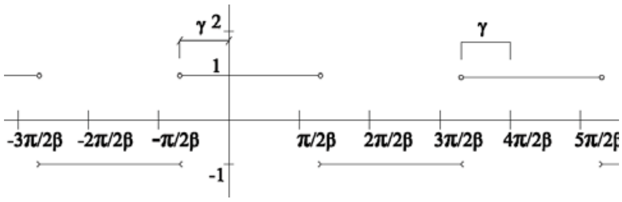


Fig. 5: The $ang_y(\beta(x+\gamma))$ waveform.

In fact:

$$ang_y(x) = \begin{cases} +1 & for\ \ \ \ 2K\pi \leq x \leq (2K+1)\pi \\ -1 & for\ \ (2K+1)\pi < x < (2K+2)\pi \end{cases}$$

For x going from -∞ to +∞ the sign of the function changes into two values +1 and -1 only,
For $x = K\pi$, it changes from:

$$\begin{cases} -1\ to +1 & for\ \ x = 2K\pi \\ +1\ to -1 & for\ x = \pi + 2K\pi = (2K+1)\pi \end{cases}$$

- Particular case: For $\beta = 1$ and $\gamma = 0$, the expression (4) becomes:

$$ang_y(x) = \begin{cases} +1 & for\ \ \sin(x) \geq 0 \\ -1 & for\ \ \sin(x) < 0 \end{cases} \quad (5)$$

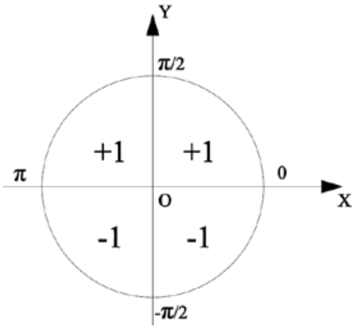


Fig. 6: The $ang_y(x)$ on the unit circle.

2.2.1 Application of the Angular function $ang_y(x)$ in the elliptical trigonometry

The Angular function $ang_y(x)$ has a huge importance in the elliptical trigonometry [4],[5] in which it helps to formulate many equations that can produce huge number of different signals by varying some parameters, the author choose one function as following in order to give an example about the importance of $ang_y(x)$:

The Absolute Elliptic *Mar* which is defined by the author in the published papers [4],[5] has the following form:

$$\bar{E}mar_{i,b}(x) = Emar_b(x) \cdot \left(ang_y(x)\right)^i \qquad (6)$$

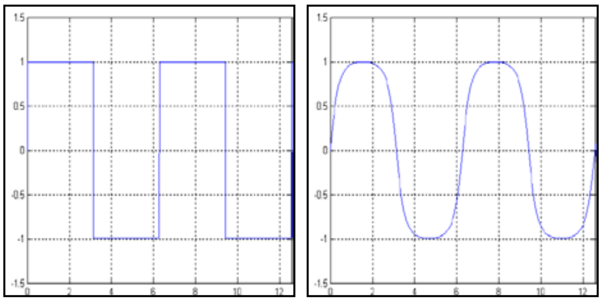
With

$$Emar_b(x) = \frac{a}{b} \frac{\text{Cter}\,(x) ang_x(x)}{\sqrt{1+\left(\frac{a}{b}\text{Cter}\,(x)\right)^2}} \qquad (7)$$

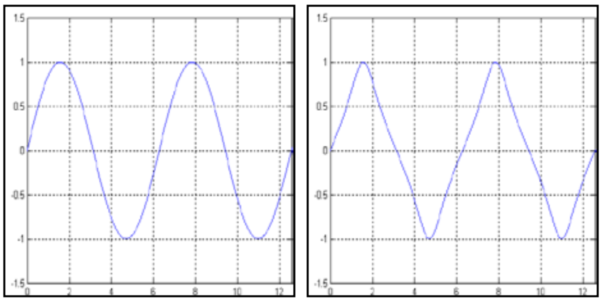
Similar to the Absolute Elliptic *Jes*, the Absolute Elliptic *Mar* is a powerful function that can produce more than 14 different signals by varying only two parameters i and b. Similar to the sine function in the traditional trigonometry, the Absolute Elliptical *Mar* is more general than the precedent.

• Multi form signals made by $\bar{E}mar_{i,b}(x)$:

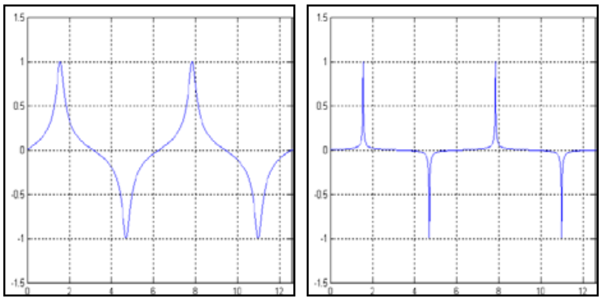
Figures 7 and 8 represent multi form signals obtained by varying two parameters (i and b). For the figures 7.a to 7.f the value of $i = 2$, for the figures 8.a to 8.f the value of $i = 1$.



a) $i = 2; b = 0.001$ b) $i = 2; b = 0.4$

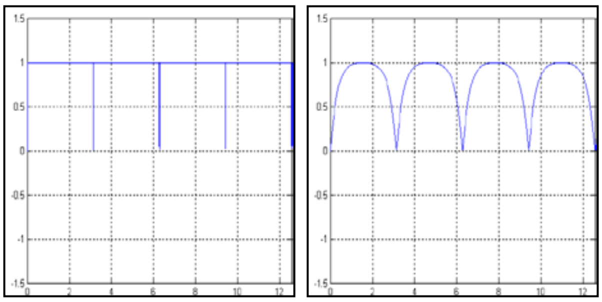


c) $i = 2; b = 1$ d) $i = 2; b = \sqrt{3}$

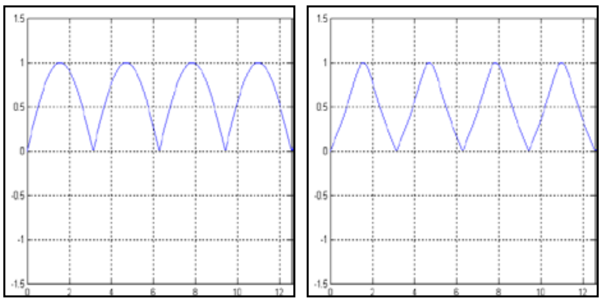


e) $i = 2; b = 6$ f) $i = 2; b = 100$

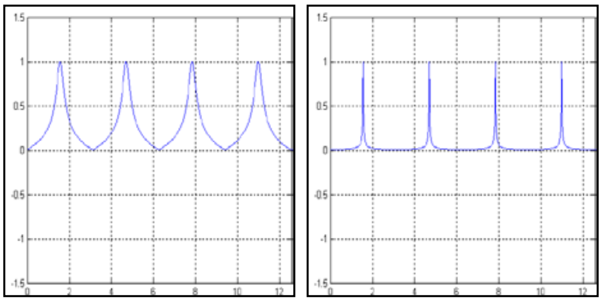
Fig. 7: multi form signals of the function $\bar{E}mar_{i,b}(x)$ for $i = 2$ and for different values of $b > 0$.



a) $i = 1; b = 0.001$ b) $i = 1; b = 0.4$



c) $i = 1; b = 1$ d) $i = 1; b = \sqrt{3}$



e) $i = 1; b = 6$ f) $i = 1; b = 100$

Fig. 8: multi form signals of the function $\bar{E}mar_{i,b}(x)$ for $i = 1$ and for different values of $b > 0$.

Important signals obtained using this function:
Impulse train with positive and negative part, elliptic deflated, quasi-triangular, sinusoidal, elliptical swollen, square signal, rectangular signal, impulse train (positive part only), rectified elliptic deflated, saw signal, rectified elliptical swollen, continuous signal…

These types of signals are widely used in power electronics, electrical generator and in transmission of analog signals [21].

2.3 Angular function $ang_\alpha(x)$

α (called firing angle) represents the angle width of the positive part in a period. In this case, one can vary the width of the positive and the negative part by varying only α. The firing angle must be positive.

$$ang_\alpha\big(\beta(x+\gamma)\big) = \begin{cases} +1 \ \ for \\ (2K\pi-\alpha)/\beta-\gamma \le x \le (2K\pi+\alpha)/\beta-\gamma \\ \\ -1 \ \ for \\ (2K\pi+\alpha)/\beta-\gamma < x < (2(K+1)\pi-\alpha)/\beta-\gamma \end{cases} \quad (8)$$

With:
β is the frequency of the function.
γ is the translation of the function on the axis (ox).
x is the a variable parameter $x \in]-\infty;+\infty[$
α is the firing angle of the function ($\alpha \ge 0$).

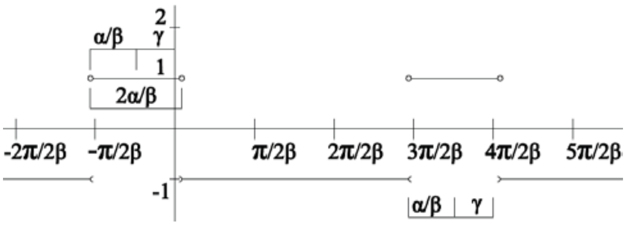


Fig. 9: The $ang_\alpha(\beta(x+\gamma))$ waveform.

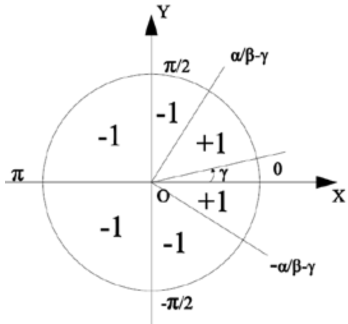


Fig. 10: The $ang_\alpha(\beta(x+\gamma))$ on the unit circle for $\beta = 1, \alpha = \frac{\pi}{4}$ and $\gamma = -13^0$.

• Particular cases:

1-For $\alpha = \pi/2$,

$\Rightarrow ang_{\frac{\pi}{2}}\big(\beta(x+\gamma)\big) = ang_x(\beta(x+\gamma))$. In fact,

$$ang_{\frac{\pi}{2}}\big(\beta(x+\gamma)\big) = \begin{cases} +1 \ \ for \\ (2K\pi-\pi/2)/\beta-\gamma \le x \le (2K\pi+\pi/2)/\beta-\gamma \\ -1 \ \ for \\ (2K\pi+\pi/2)/\beta-\gamma < x < (2(K+1)\pi-\pi/2)/\beta-\gamma \end{cases}$$

$$= \begin{cases} +1 \ \ for \ (4K-1)\frac{\pi}{2\beta}-\gamma \le x \le (4K+1)\frac{\pi}{2\beta}-\gamma \\ -1 \ \ for \ (4K+1)\frac{\pi}{2\beta}-\gamma \ < x < (4K+3)\frac{\pi}{2\beta}-\gamma \end{cases}$$

$$\Rightarrow ang_{\frac{\pi}{2}}\big(\beta(x+\gamma)\big) = ang_x(\beta(x+\gamma))$$

2-For $\alpha = \pi/2$ and $\gamma' = -\frac{\pi}{2\beta}+\gamma$,

$\Rightarrow ang_{\frac{\pi}{2}}\big(\beta(x+\gamma')\big) = ang_y(\beta(x+\gamma))$. In fact,

$$ang_{\frac{\pi}{2}}\big(\beta(x+\gamma')\big) = \begin{cases} +1 \ \ for \ \ (4K-1)\frac{\pi}{2\beta}-\gamma' \le x \le (4K+1)\frac{\pi}{2\beta}-\gamma' \\ -1 \ \ for \ \ (4K+1)\frac{\pi}{2\beta}-\gamma' \ < x < (4K+3)\frac{\pi}{2\beta}-\gamma' \end{cases}$$

$$= \begin{cases} +1 \ \ for \qquad (4K)\frac{\pi}{2\beta}-\gamma \le x \le (4K+2)\frac{\pi}{2\beta}-\gamma \\ -1 \ \ for \ \ (4K+2)\frac{\pi}{2\beta}-\gamma \ < x < (4K+4)\frac{\pi}{2\beta}-\gamma \end{cases}$$

$$= \begin{cases} +1 \ \ for \qquad 2K\pi/\beta-\gamma \le x \le (2K+1)\pi/\beta-\gamma \\ -1 \ \ for \ \ (2K+1)\pi/\beta-\gamma < x < (2K+2)\pi/\beta-\gamma \end{cases}$$

$$\Rightarrow ang_{\frac{\pi}{2}}\big(\beta(x+\gamma')\big) = ang_{\frac{\pi}{2}}\left(\beta(x+\gamma)-\frac{\pi}{2}\right) = ang_x\left(\beta(x+\gamma)-\frac{\pi}{2}\right) = ang_y(\beta(x+\gamma))$$

3-for $\beta = 1, k = 0$ and $\gamma = 0$, therefore,

$$ang_\alpha(x) = \begin{cases} +1 \ \ for \quad -\alpha \le x \le +\alpha \\ -1 \ \ for \quad +\alpha < x < 2\pi-\alpha \end{cases}$$

2.3.1 Application of the Angular function $ang_\alpha(x)$ in the Rectangular trigonometry

The Angular function $ang_\alpha(x)$ has a huge importance in the Rectangular trigonometry [3] in which it helps to formulate many equations that can produce huge number of different signals by varying some parameters, the author choose one function as following in order to give an example about the importance of $ang_\alpha(x)$:

The Absolute Rectangular *Jes* which is defined by the author in the published paper [3] has the following form:

• Expression of the Absolute Rectangular *Jes*:

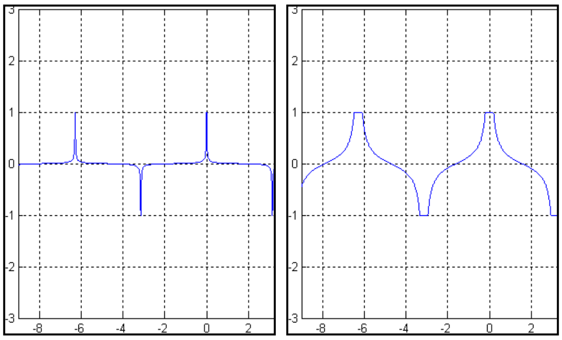
$$\bar{R}jes_{i,b}(x) = Rjes_b(x)\cdot\left(ang_x(x)\right)^i \qquad (9)$$

With $Rjes_b(x) =$

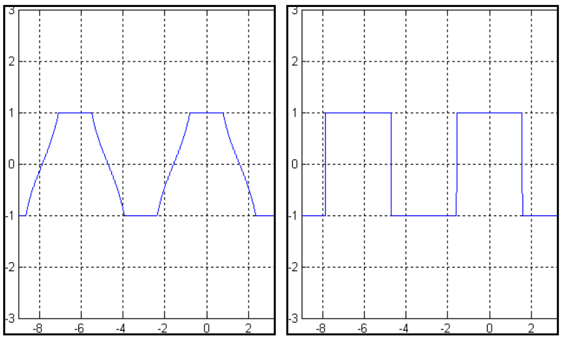
$$ang_y(x+\beta)\cdot\left(\frac{a}{b}\mathrm{Cter}(x)\right)^{\frac{\mathrm{ang}_\beta(\mathrm{x})+\mathrm{ang}_\beta(\mathrm{x}-\pi)}{2}} \qquad (10)$$

• Multi form signals made by $\bar{R}jes_{i,b}(x)$:

Figures 11 and 12 represent multi form signals obtained by varying two parameters (i and b). For the figures 11.a to 11.d the value of $i = 2$, for the figures 12.a to 12.d the value of $i = 1$.

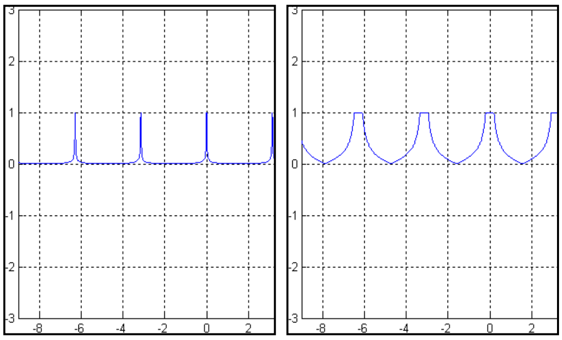


a) $i = 2, b = 0.01$ b) $i = 2, b = 0.2$

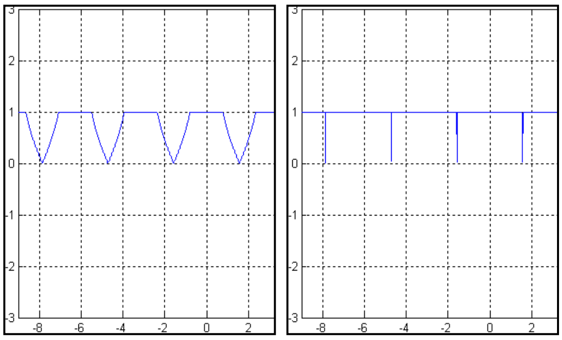


c) $i = 2, b = 1$ d) $i = 2, b = 100$

Fig. 11: multi form signals of the function $\bar{R}jes_{i,b}(x)$ for $i = 2$ and for different values of $b > 0$.



a) $i = 1, b = 0.01$ b) $i = 1, b = 0.2$



c) $i = 1, b = 1$ d) $i = 1, b = 100$

Fig. 12: multi form signals of the function $\bar{R}jes_{i,b}(x)$ for $i = 1$ and for different values of $b > 0$.

Important signals obtained using this function: Alternate impulse train (positive and negative part), Impulse train (positive part only), square, rectangle, saturated saw signal, saturated triangle signal, continuous signal …

These types of signals are widely used in power electronics, electrical generator and in transmission of analog signals [21].

2.4 Angular function $ang_{|\alpha|}(x)$

The function $ang_{|\alpha|}(x)$ has the same principle as $ang_\alpha(x)$, but the difference is that the firing angle is presented in an absolute value, which is mean, the positive part is presented between 0 and α ($0 \le x \le \alpha$) (figure 13). By varying α, the width of the positive and negative parts of a period is controlled. The firing angle α must be positive.

$$ang_{|\alpha|}(x) = \begin{cases} +1 \ \ for \ \ \ 0 \le x \le \alpha \\ -1 \ \ for \ \ \alpha < x < 2\pi \end{cases} \quad (11)$$

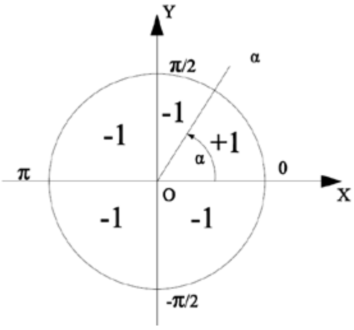


Fig. 13: The $ang_{|\alpha|}(x)$ on the unit circle.

The general definition of $ang_{|\alpha|}(x)$:

$$ang_{|\alpha|}\big(\beta(x+\gamma)\big) = \begin{cases} +1 \ \ for \ \ \ \ \ \ \ \ 2K\pi/\beta - \gamma \le x \le (2K\pi+\alpha)/\beta - \gamma \\ -1 \ \ for \ \ (2K\pi+\alpha)/\beta - \gamma < x < 2(K+1)\pi/\beta - \gamma \end{cases} \quad (12)$$

With $0 \le \alpha < 2\pi/\beta$

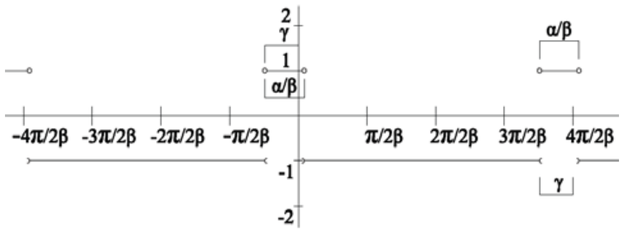


Fig. 14: The $ang_{|\alpha|}\big(\beta(x+\gamma)\big)$ waveform.

It is important to make sure that angles $\alpha, x \ and \ \gamma$ have the same unit for example degree, radian or square angle… In general, the radian unit is taken.
In the same way $ang_{|f|}(x)$ is defined, which f can be $x, y \ or \ \alpha$.

2.5 Angular function $ang_{\Sigma|\alpha|}(x)$

The general and the most important form of the angular function is $ang_{\Sigma|\alpha_k|}(\beta_k(x+\gamma))$, with α_k is a variable firing angle and β_k is a variable frequency, this function describe the Pulse Width Modulation in power electronic and it can be transformed to any other angular function.

$$ang_{\Sigma|\alpha_k|}(\beta_k(x+\gamma)) = \begin{cases} +1 \ \ for \ \ A \le x \le B \\ -1 \ \ for \ \ B < x < C \end{cases} \quad (13)$$

With:

- $A = \left(sign(k).\sum_{i=1-u(k)}^{|k|-u(k)} T_{sign(k).i} \right) - \gamma$
- $B = \left(sign(k).\sum_{i=1-u(k)}^{|k|-u(k)} T_{sign(k).i} \right) + \alpha_k - \gamma$
- $C = \left(sign(k+1).\sum_{i=1-u(k+1)}^{|k+1|-u(k+1)} T_{sign(k+1).i} \right) - \gamma$
- $T_\mu = 2\pi/\beta_\mu$
- $k \in \mathbb{Z}$
- $\alpha_k = f(k)$ with $0 \le \alpha_k < 2\pi/\beta_k$,
E.g.: $\alpha_k = |k + 3\sin(k)|$.
- $\beta_k = f(k) > 0$, e.g.: $\beta_k = |2k^2 + 4|$.
- $u(k) = \begin{cases} 1 \ \ for \ k \ge 0 \\ 0 \ \ for \ k < 0 \end{cases}$

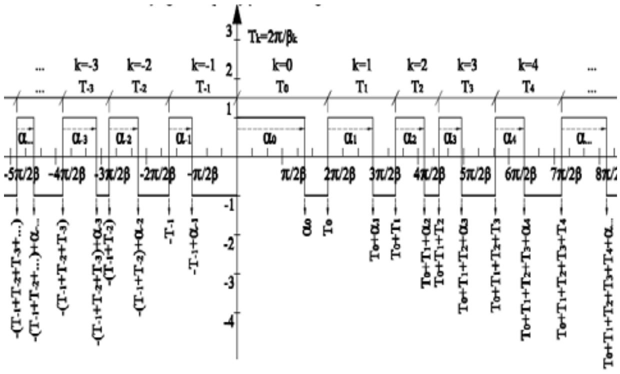


Fig. 15: The $ang_{\Sigma|\alpha_k|}(\beta_k(x+\gamma))$ waveform.

- Particular cases:

1-For $\beta_k = \beta$. Thus, $T_k = T = \frac{2\pi}{\beta_k} = \frac{2\pi}{\beta}$. Therefore,
$ang_{\Sigma|\alpha_k|}(\beta_k(x+\gamma)) = ang_{|\alpha_k|}(\beta(x+\gamma))$.

2- For $\beta_k = \beta$ and $\alpha_k = \alpha$. Therefore,
$ang_{\Sigma|\alpha_k|}(\beta_k(x+\gamma)) = ang_{|\alpha|}(\beta(x+\gamma))$

3-For $\beta_k = \beta$ and $\alpha_k = \alpha = \frac{\pi}{\beta}$. Therefore,
$ang_{\Sigma|\alpha_k|}(\beta_k(x+\gamma)) = ang_y(\beta(x+\gamma))$

4- For $\beta_k = \beta$, $\alpha_k = \alpha = \frac{\pi}{\beta}$ and $\gamma' = \frac{\pi}{2\beta} + \gamma$.
Therefore,
$ang_{\Sigma|\alpha_k|}(\beta_k(x+\gamma')) = ang_x(\beta(x+\gamma))$

2.5.1 Example using $ang_{\Sigma|\alpha|}(x)$

For $\beta_k = k^2 + 1$, $\alpha_k = \pi/(k^2+3)$ and $\gamma = 0$. Therefore, the following figure is obtained.

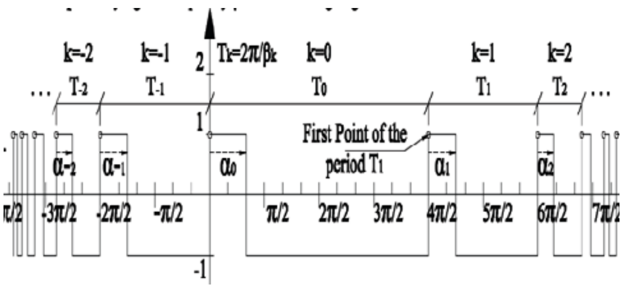


Fig. 16: The $ang_{\Sigma|\alpha_k|}(\beta_k(x+\gamma))$ waveform.

2.6 Remarkable functions

- $ang_f(\alpha)=\frac{1}{ang_f(\alpha)} \Rightarrow \left(ang_f(\alpha)\right)^2=1$
- $ang_x(x)=2(\sum_{k\in\mathbb{Z}} rect_\pi(x-2k\pi))-1$
- $ang_y(x)=2(\sum_{k\in\mathbb{Z}} rect_\pi(x-(\frac{\pi}{2}+2k\pi)))-1$
- $ang_x(x).\cos(x)=|\cos(x)|$
- $ang_y(x).\sin(x)=|\sin(x)|$
- $ang_y(x)=ang_x\left(x-\frac{\pi}{2}\right)$
- $ang_{\pi/2}(x)=ang_x(x)$
- $ang_{\pi/2}(x-\pi/2)=ang_y(x)$
- $ang_\alpha(x)=\frac{\left[1+ang_x\left(x+\frac{\pi}{2}-\alpha\right)\right]\cdot\left[1+ang_y(x+\alpha)\right]}{2}-1$

3 Derivation of the angular functions

In this section, the derivations of some angular functions are represented. In order to make the derivation of the angular functions, it is necessary to define some useful definitions:

- Derivation of $f(x)$:

$$f(x)'=\frac{df(x)}{dx}=\lim_{x_0\to 0}\frac{f(x+x_0)-f(x)}{(x+x_0)-x} \quad (14)$$

- Unit step function:

$$U(x)=\begin{cases}+1 \; for \; x\geq 0\\ 0 \; for \; x<0\end{cases}$$ (11). Thus,

$$U(x)'=\lim_{x_0\to 0}\frac{U(x+x_0)-U(x)}{x_0}=\frac{1}{x_0}=\delta(x) \quad (15)$$

With $\delta(x)$ is the Dirac pulse function, in fact,

$$\delta(x)=\begin{cases}\delta(x-x_0)=0 \; for \; x\neq x_0\\ \int_{-\infty}^{+\infty}\delta(x-x_0)dx=1\end{cases} \quad (16)$$

3.1 Derivation of $\boldsymbol{ang_x(\beta(x+\gamma))}$

$$ang_x\big(\beta(x+\gamma)\big)'=$$
$$2(-1)^{K+1}.\delta\left(x-\left(\frac{(2K+1)\pi}{2\beta}-\gamma\right)\right) \quad (17)$$

3.2 Derivation of $\boldsymbol{ang_y(\beta(x+\gamma))}$

$$ang_y\big(\beta(x+\gamma)\big)'=$$
$$2(-1)^{K}.\delta\left(x-\left(\frac{K\pi}{\beta}-\gamma\right)\right) \quad (18)$$

3.3 Derivation of $\boldsymbol{ang_\alpha(\beta(x+\gamma))}$

$$ang_\alpha\big(\beta(x+\gamma)\big)'=$$
$$\begin{cases}2(-1)^{2K}.\delta\left(x-\left(\frac{2K\pi-\alpha}{\beta}-\gamma\right)\right) & (19)\\ 2(-1)^{2K+1}.\delta\left(x-\left(\frac{2K\pi+\alpha}{2\beta}-\gamma\right)\right) & (20)\end{cases} \text{ with } K\epsilon\mathbb{Z}$$

The expression (19) gives $+2\delta$, when the value of $ang_\alpha(\beta(x+\gamma))$ pass from -1 to +1.
The expression (20) gives -2δ, when the value of $ang_\alpha(\beta(x+\gamma))$ pass from +1 to -1.

By using the definition of the *InfoMath* function that is defined in the section 4, the two expressions (19) and (20) can be written in one expression as the following:

$$ang_\alpha\big(\beta(x+\gamma)\big)'=$$
$$2\frac{K}{|K|}(-1)^K.\delta\left(x-\left(\frac{\left(K+\underline{K;I;\left(-\frac{K}{|K|}\right)}\right)\pi-\alpha\frac{K}{|K|}(-1)^K}{\beta}-\gamma\right)\right) (21)$$

With $K\epsilon\mathbb{Z}^* \; or \; \mathbb{Z}-\{0\}$ (only for this case).

And $\underline{K;I;\left(-\frac{K}{|K|}\right)}=\begin{cases}-\frac{K}{|K|} \; if \; k \; is \; an \; odd \; number\\ +1 \; if \; k \; is \; an \; even \; number\end{cases}$

3.4 Derivation of $\boldsymbol{ang_{|\alpha|}(\beta(x+\gamma))}$

$$ang_{|\alpha|}\big(\beta(x+\gamma)\big)'=$$
$$\begin{cases}2(-1)^{2K}.\delta\left(x-\left(\frac{2K\pi}{\beta}-\gamma\right)\right) & (22)\\ 2(-1)^{2K+1}.\delta\left(x-\left(\frac{2K\pi+\alpha}{2\beta}-\gamma\right)\right) & (23)\end{cases} \text{ With } K\epsilon\mathbb{Z}$$

The expression (22) gives $+2\delta$, when the value of $ang_{|\alpha|}(\beta(x+\gamma))$ pass from -1 to +1.

The expression (23) gives -2δ, when the value of $ang_{|\alpha|}(\beta(x+\gamma))$ pass from +1 to -1.

By using the definition of the *InfoMath* function that is defined in the section 4, the two expressions (22) and (23) can be written in one expression as the following:

$$ang_{|\alpha|}(\beta(x+\gamma))' = 2(-1)^K.\delta\left(x-\left(\frac{\left(K+\underline{K;I;(-1)}\right)\pi+\underline{K;p;(0)}.\alpha}{\beta}-\gamma\right)\right) \quad (24)$$

With:

- $K \epsilon \mathbb{Z}$
- $\underline{K;I;(-1)} = \begin{cases} -1 \ if \ k \ is \ an \ odd \ number \\ +1 \ if \ k \ is \ an \ even \ number \end{cases}$
- $\underline{K;P;(0)} = \begin{cases} 0 \ if \ k \ is \ an \ even \ number \\ +1 \ if \ k \ is \ an \ odd \ number \end{cases}$

4 A survey on the InfoMath function

The InfoMath function is an original function introduced by the author in the mathematical domain. It has an exceedingly importance in mathematics and in all scientific domains.

The InfoMath Function can describe an infinite number of functions and can unify many expressions or formulae into only one expression or formula. It is introduced in the mathematical domain to facilitate and reduce many expressions and forms to a simple expression that has the same characteristics as the precedents.

The complete study of the InfoMath function will be submitted later. In this paper, only a brief survey is introduced in order to understand how the expressions (21) and (24) works.

4.1 Structure of the InfoMath function

The InfoMath function is composed of 3 parts: $input$, $condition$ and $output$. By varying the parameters of these parts, one can describe an infinite number of forms and functions. E.g: one can describe a square waveform, rectangular function, triangular function, pulse of Dirac, Hysteresis curve, sign function…

• The $input$ can be a constant number, scalar, parameter, expression, vector, variable, matrix … The same for the $output$.

• The condition part is the logic part of the function and any type of operator can be used (logic, relational, intervals, relation with groups or elements, domain of the number…)

The InfoMath function can be denoted as the following form ($\underline{inputs; conditions; output}$)

The InfoMath function $\underline{K;I;\left(-\frac{K}{|K|}\right)}$ that appears in the expression (21), can be read as the following:

• K is the input part, it is a variable parameter.

• I is the conditional part, it means that if K is an odd number (Impair in French).

• $-\frac{K}{|K|}$ is the output if the condition is verified (K is an odd number). Otherwise, the default output is "1" (if K is not an odd number).

The InfoMath function $\underline{K;P;(0)}$ that appears in the expression (24), can be read as the following:

• K is the input part, it is a variable parameter.

• P is the conditional part, it means that if K is an even number (Pair in French).

• 0 is the output if the condition is verified (K is an even number). Otherwise, the default output is "1" (if K is not an even number).

4.2 A simple example using the InfoMath function

• $y = x + x.\underline{x\,;>5;(x)}$ (25)

The InfoMath function $\underline{x\,;>5;(x)}$ can be read as follows:

If x is >5 the condition is verified, the output will be (x), $\Rightarrow y = x + x.(x) = x + x^2$. Otherwise, if the condition is not verified the output will be $(1) \Rightarrow y = x + x.(1) = 2x.$

Therefore;

$$y = x + x.\underline{x\,;>5;(x)} \Rightarrow \ y = \begin{cases} 2x \quad if \ x \leq 5 \\ x + x^2 \ \ if \ x > 5 \end{cases}$$

The InfoMath function is one of the most complicated functions introduced in the mathematical domain. In fact, the number of inputs is unlimited, the same for the number of operators and outputs. In another hand, one can mixed the type of inputs, conditions or outputs, for example the input can be at the same time a vector or a matrix or a parameter.

5 Definition of the step-rectangular waveform "n" order

The step-rectangular waveform "n" order is a periodic signal that contains "n" complete rectangles in a semi-period or 2n complete rectangles in a period. The formula of this signal can be written using the angular function $ang_x(\beta(x+\gamma))$:

$\sum_{\gamma=0}^{n-1} \frac{ang_x\left(\beta\left(x-\frac{\gamma}{n}\frac{\pi}{\beta}\right)\right)}{n}$; with $0 \le \frac{\gamma}{n}\frac{\pi}{\beta} < \frac{\pi}{\beta}$ to avoid overlapping.

With $ang_x\left(\beta\left(x-\frac{\gamma}{n}\frac{\pi}{\beta}\right)\right) =$

$$\begin{cases} +1 \;\; for \;\; (4K-1)\frac{\pi}{2\beta}+\frac{\gamma}{n}\frac{\pi}{\beta} \le x \le (4K+1)\frac{\pi}{2\beta}+\frac{\gamma}{n}\frac{\pi}{\beta} \\ -1 \;\; for \;\; (4K+1)\frac{\pi}{2\beta}+\frac{\gamma}{n}\frac{\pi}{\beta} < x < (4K+3)\frac{\pi}{2\beta}+\frac{\gamma}{n}\frac{\pi}{\beta} \end{cases} \quad (26)$$

Particular cases:

1- Step-rectangular waveform first order (n=1)

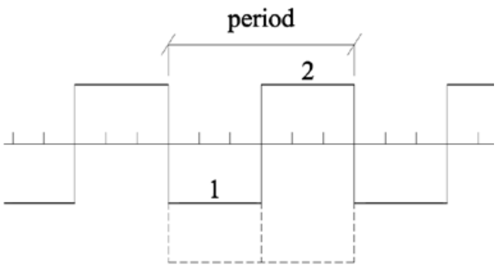


Fig. 17: number of rectangles in the Step-rectangular waveform first order (n=1).

2- Step-rectangular waveform third order (n=3)

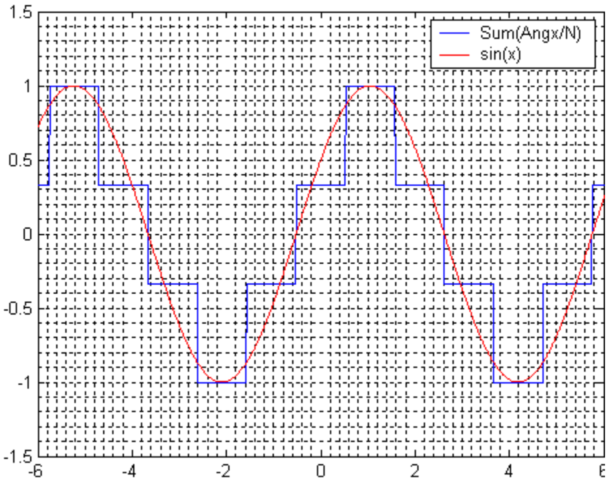


Fig. 18: comparing the Step-rectangular waveform third order and the sine waveform using Matlab.

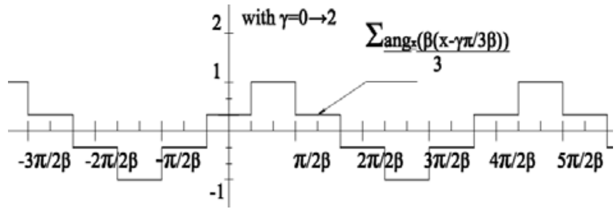


Fig. 19: Step-rectangular waveform third order (n=3).

3- Step-rectangular waveform sixth order (n=6)

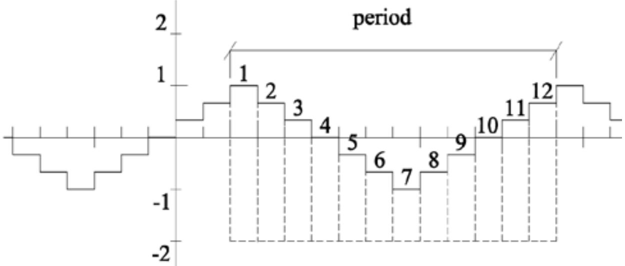


Fig. 20: number of rectangles in a period of the Step-rectangular waveform sixth order (n=6).

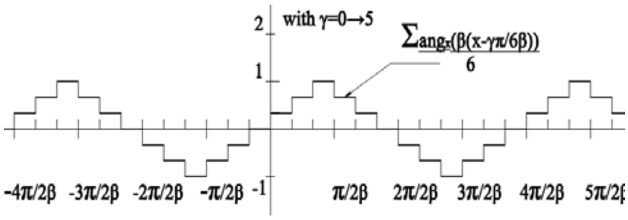


Fig. 21: Step-rectangular waveform sixth order (n=6).

4- Step-rectangular waveform "100" order (n=100)

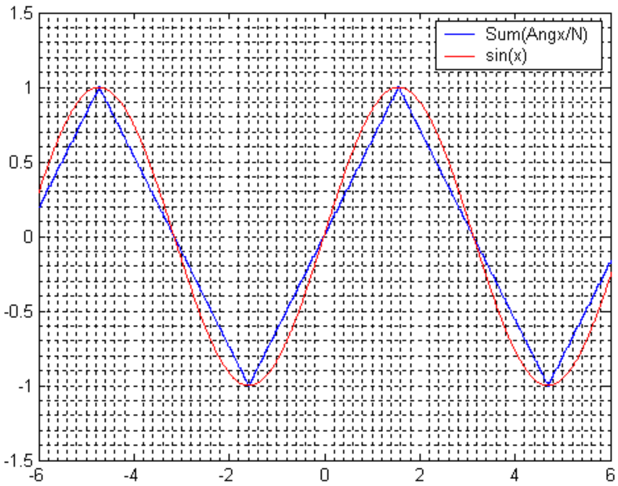


Fig. 22: comparing the Step-rectangular waveform "100" order and the sine waveform using Matlab.

When the number of order "n" increase to the infinite, the shape of the function takes a triangular waveform.

6 A survey on the applications of the Angular Functions in the electrical engineering domain

These functions are easily programmed and simulated using software as Matlab, Simulink Labview … In other hand, many integrated circuits can be used to describe these functions such as ATMEL AT40K, MOTOROLA TL494. It can be used in many applications such as controlling the speed of the motor, generating a PWM signal, controlling the firing angle of a sinusoidal signal…

In trigonometry topic, and particularly the Elliptical Trigonometry is based on the angular functions, in which one elliptical trigonometric function can produce more than 14 different signals [1],[2]. These functions have an extremely importance in electronics in which only one electronic circuit can produce many periodic signals by varying some parameters, this is not the case of any other existing circuit.

In power electronics, the angular functions will be widely used. Particularly, many signals can be produced by using the angular functions. The following examples can give an idea about how the angular functions can be used. For the example 1, Refer to "Power electronics" book, third edition, Cyril W. Lander, pp. 50-52). For the example 2, Refer to "Power electronics" book, third edition, Cyril W. Lander, pp. 53-54).

-Example 1: A single phase bridge (or double way) fully controlled circuit using four thyristors can be described by using the following expression:

$Y_1 = \sin(2\beta x) \, . \, ang_y(2\beta(x+\gamma))$ with $\gamma \leq 0$

This expression describes the waveform of the load voltage that contains resistance and inductance. By varying γ, the mean voltage load can be controlled. For $\gamma = 0$, the mean voltage is maximum. For $\gamma < 0$, the mean voltage decrease (figure 23).

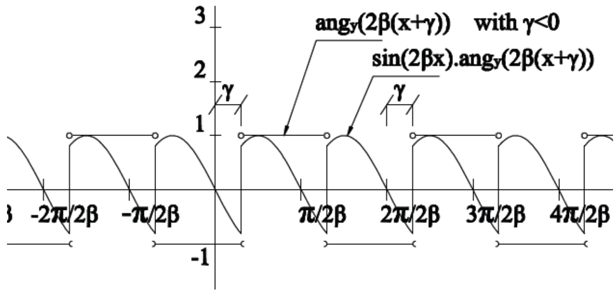


Fig. 23: Y_1 describes the waveform of the single phase bridge fully controlled circuit.

In the same way, the waveform of the thyristor voltage v_{T1} on the thyristor T1 can be described by using the following expression:

$Y_2 = \mathrm{Sin}(2\beta x) \, . \, \frac{1 - ang_y(2\beta(x+\gamma))}{2}$

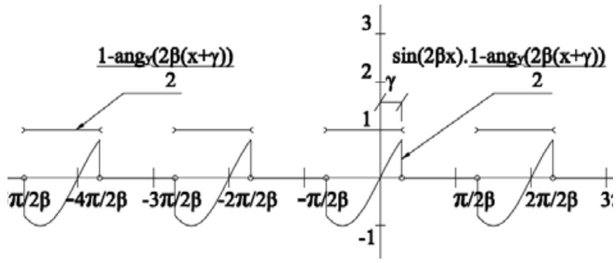


Fig. 24: Y_2 describes the waveform of the thyristor voltage v_{T1} on the thyristor T1.

-Example 2: a single phase bridge (or double way) half-controlled connection circuit using two thyristors, two diodes and a commutating diode can be described using the following expression:

$Y_3 = \sin\,(2\beta x).\, ang_y(2\beta x).\frac{1 - ang_{|\alpha|}(4\beta(x+\gamma))}{2}$

This expression describes the waveform of the load voltage that contains resistance and inductance. The mean voltage load can be varied by varying the firing angle α.

For $\alpha = 0$, the mean voltage is maximum. For $\alpha > 0$, the mean voltage decrease. $\gamma = 0$ for this example.

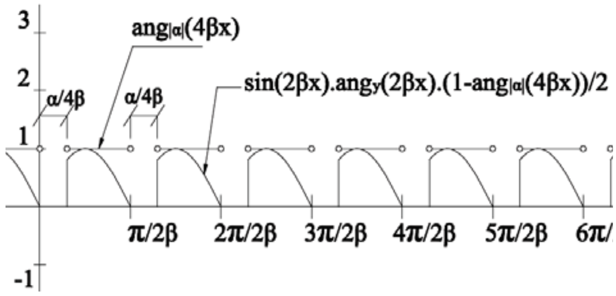


Fig. 25: Y_3 describes the waveform of a single phase bridge half-controlled that is treated in example 2.

7 Conclusion

In this paper, an original study in the mathematical field is introduced by the author. And some applications in the engineering domains are presented in order to emphasize the importance of these new functions. In this paper, many functions are defined as ang_x , ang_y , ang_α , $ang_{|\alpha|}$ and $ang_{\Sigma|\alpha_k|}$. For each function, an example is given in order to facilitate the comprehension of these new functions. These new functions have enormous importance in electronics and in signal theory in which the Angular functions can create new functions as we have seen it in the previous sections and moreover, the angular functions are the bases of the elliptical and rectangular trigonometry which are introduced by the author and are also published [1],[2],[3],[4],[5]. The importance of the angular functions is not limited only in the electrical and electronic domains. They can be used as any other functions in any domain.

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