

$$\begin{cases} w_1'' = f_*(t, w_1, w_1'), t \in J_1, \\ w_1(t_1^+) = I_{21}(\eta'(t_1)) + C_1(w_0'(t_1) - \eta'(t_1)) + w_0(t_1), \\ w_1'(t_1^+) = I_{31}((B\eta)(t_1), (F\eta)(t_1), \eta(t_1), \eta'(t_1)) \\ \quad - L_1(w_0(t_1) - \eta(t_1)) \\ \quad - L_1^*(w_0'(t_1) - \eta'(t_1)) + w_0'(t_1) \end{cases} \quad (35)$$

Again, we want to prove that linear differential equation for any $i, (i = 1, 2, \dots, m)$

$$\begin{cases} u'' = f_*(t, u, u'), t \in J_i, \\ u(t_i^+) = I_{2i}(\eta'(t_i)) + C_i(w_{i-1}'(t_i) - \eta'(t_i)) + w_{i-1}(t_i), \\ u'(t_i^+) = I_{3i}((B\eta)(t_i), (F\eta)(t_i), \eta(t_i), \eta'(t_i)) \\ \quad - L_i(w_{i-1}(t_i) - \eta(t_i)) \\ \quad - L_i^*(w_{i-1}'(t_i) - \eta'(t_i)) + w_{i-1}'(t_i), \end{cases}$$

has a unique solution

$w_i \in PC^1[J_i, E] \cap C^2[(t_i, t_{i+1}), E]$ such that

$$\begin{cases} w_i'' = f_*(t, w_i, w_i'), t \in J_i, \\ w_i(t_i^+) = I_{2i}(\eta'(t_i)) + C_i(w_{i-1}'(t_i) - \eta'(t_i)) + w_{i-1}(t_i), \\ w_i'(t_i^+) = I_{3i}((B\eta)(t_i), (F\eta)(t_i), \eta(t_i), \eta'(t_i)) \\ \quad - L_i(w_{i-1}(t_i) - \eta(t_i)) - L_i^*(w_{i-1}'(t_i) - \eta'(t_i)) \\ \quad + w_{i-1}'(t_i). \end{cases} \quad (36)$$

Let

$$u_\eta(t) = \begin{cases} w_0(t), t \in J_0, \\ w_1(t), t \in J_1, \\ \dots\dots\dots \\ w_m(t), t \in J_m. \end{cases} \quad (37)$$

Combing (31) and (35), (36), (37), we have $u_\eta \in PC^1[J, E] \cap C^2[J', E]$ is a unique solution of IVP (22).

Putting $u_\eta = A\eta$, Then

$$A: [u_0, v_0] \rightarrow PC^1[J, E] \cap C^2[J', E].$$

Next we prove two cases:

Case (1): $u_0 \leq Au_0, u_0' \leq (Au_0)'$,

$$Av_0 \leq v_0, (Av_0)' \leq v_0'.$$

Case (2): if $\eta_1, \eta_2 \in [u_0, v_0]$ and

$$\eta_1 \leq \eta_2, \eta_1' \leq \eta_2', \text{ then } A\eta_1 \leq A\eta_2, (A\eta_1)' \leq (A\eta_2)'.$$

First, consider case (1).

Put $u_1 = Au_0, p = u_0 - u_1$. By (22), we have

$$\begin{aligned} u_1''(t) &= -M_1(t)u_1(t) - M_2(t)u_1'(t) + M_1(t)u_0(t) \\ &\quad + M_2(t)u_0'(t) + f(t, (Bu_0)(t), (Fu_0)(t), \\ &\quad u_0(t), u_0'(t), (TBu_0)(t), (SBU_0)(t)), \forall t \in J, t \neq t_k, \end{aligned}$$

$$\Delta u_1|_{t=t_k} = I_{2k}(u_0'(t_k)) + C_k(u_1'(t_k) - u_0'(t_k)),$$

$$\begin{aligned} \Delta u_1'|_{t=t_k} &= I_{3k}((Bu_0)(t_k), (Fu_0)(t_k), u_0(t_k), \\ &\quad u_0(t_k), u_0'(t_k)) - L_k(u_1(t_k) - u_0(t_k)) \\ &\quad - L_k^*(u_1'(t_k) - u_0'(t_k)) \quad (k = 1, 2, \dots, m), \end{aligned}$$

$$u_1(0) = x_2^*, u_1'(0) = x_3^*.$$

Moreover, by (H_1) we have

$$\begin{cases} p''(t) = u_0''(t) - u_1''(t) \\ \quad \leq -M_1(t)p(t) - M_2(t)p'(t), \forall t \in J, t \neq t_k, \\ \Delta p|_{t=t_k} = \Delta u_0|_{t=t_k} - \Delta u_1|_{t=t_k} = C_k p'(t_k), \\ \Delta p'|_{t=t_k} = \Delta u_0'|_{t=t_k} - \Delta u_1'|_{t=t_k} \\ \quad \leq -L_k p(t_k) - L_k^* p'(t_k) \quad (k = 1, 2, \dots, m), \\ p'(0) = u_0'(0) - u_1'(0) = u_0'(0) - x_3^* \\ \quad \leq u_0'(0) - x_2^* = p(0) \leq \theta. \end{cases} \quad (38)$$

Hence, by Lemma 2.1 we obtain

$$p(t) \leq \theta, p'(t) \leq \theta, \forall t \in J.$$

This means that

$$u_0 \leq Au_0, u_0' \leq (Au_0)'.$$

In the same way, $Av_0 \leq v_0, (Av_0)' \leq v_0'$,

Next, consider case (2):

Let $\eta_1, \eta_2 \in [u_0, v_0]$ such that $\eta_1 \leq \eta_2, \eta_1' \leq \eta_2'$ and put $p = \lambda_1 - \lambda_2$, where $\lambda_1 = A\eta_1, \lambda_2 = A\eta_2$.

Combing (22) and (H_2) , we have

$$\begin{cases} p''(t) = \lambda_1''(t) - \lambda_2''(t) = -M_1(t)p(t) - M_2(t)p'(t) \\ \quad - (f(t, (B\eta_2)(t), (F\eta_2)(t), \eta_2(t), \eta_2'(t), \\ \quad (TB\eta_2)(t), (SB\eta_2)(t)) - f(t, (B\eta_1)(t), \\ \quad (F\eta_1)(t), \eta_1(t), \eta_1'(t), (TB\eta_1)(t), (SB\eta_1)(t))) \\ \quad + M_1(t)(\eta_2(t) - \eta_1(t)) + M_2(t)(\eta_2'(t) - \eta_1'(t)) \\ \quad \leq -M_1(t)p(t) - M_2(t)p'(t), \forall t \in J, t \neq t_k, \\ \Delta p|_{t=t_k} = \Delta \lambda_1|_{t=t_k} - \Delta \lambda_2|_{t=t_k} = I_{2k}(\eta_1'(t_k)) \\ \quad + C_k(\lambda_1'(t_k) - \eta_1'(t_k)) - I_{2k}(\eta_2'(t_k)) \\ \quad - C_k(\lambda_2'(t_k) - \eta_2'(t_k)) = C_k p'(t_k), \\ \Delta p'|_{t=t_k} = \Delta \lambda_1'|_{t=t_k} - \Delta \lambda_2'|_{t=t_k} \\ \quad = I_{3k}((B\eta_1)(t_k), (F\eta_1)(t_k), \eta_1(t_k), \eta_1'(t_k)) \\ \quad - L_k(\lambda_1(t_k) - \eta_1(t_k)) - L_k^*(\lambda_1'(t_k) - \eta_1'(t_k)) \\ \quad - I_{3k}((B\eta_2)(t_k), (F\eta_2)(t_k), \eta_2(t_k), \eta_2'(t_k)) \\ \quad + L_k(\lambda_2(t_k) - \eta_2(t_k)) + L_k^*(\lambda_2'(t_k) - \eta_2'(t_k)) \\ \quad \leq -L_k p(t_k) - L_k^* p'(t_k) \quad (k = 1, 2, \dots, m), \\ p'(0) = p(0) = \theta. \end{cases}$$

Moreover, by Lemma 2.1 we obtain

$$p(t) \leq \theta, p'(t) \leq \theta, \forall t \in J,$$

this means that

$$(A\eta_1)(t) \leq (A\eta_2)(t), (A\eta_1)'(t) \leq (A\eta_2)'(t).$$

Let

$$u_n = Au_{n-1}, v_n = Av_{n-1} (n=1, 2, \dots). \tag{39}$$

By Case (1) and Case (2), we have

$$u_0(t) \leq u_1(t) \leq \dots \leq u_n(t) \leq \dots$$

$$\leq v_n(t) \leq \dots \leq v_1(t) \leq v_0(t), \forall t \in J.$$

$$u'_0(t) \leq u'_1(t) \leq \dots \leq u'_n(t) \leq \dots$$

$$\leq v'_n(t) \leq \dots \leq v'_1(t) \leq v'_0(t), \forall t \in J. \tag{40}$$

Let

$$U = \{u_n \mid n=1, 2, \dots\}, U' = \{u'_n \mid n=1, 2, \dots\},$$

$$U(t) = \{u_n(t) \mid n=1, 2, \dots\},$$

$$U'(t) = \{u'_n(t) \mid n=1, 2, \dots\}, t \in J.$$

By normality of P and (40), then U, U' are both bounded sets in $PC[J, E]$. For any $\eta \in [u_0, v_0]$, combing (H_1) and (H_2) , we have

$$\begin{aligned} & u''_0(t) + M_1(t)u_0(t) + M_2(t)u'_0(t) \\ & \leq f(t, (Bu_0)(t), (Fu_0)(t), u_0(t), u'_0(t), (TBu_0)(t), \\ & \quad (SBu_0)(t)) + M_1(t)u_0(t) + M_2(t)u'_0(t) \\ & \leq f(t, (B\eta)(t), (F\eta)(t), \eta(t), \eta'(t), (TB\eta)(t), \\ & \quad (SB\eta)(t)) + M_1(t)\eta(t) + M_2(t)\eta'(t) \\ & \leq f(t, (Bv_0)(t), (Fv_0)(t), v_0(t), v'_0(t), (TBv_0)(t), \\ & \quad (SBv_0)(t)) + M_1(t)v_0(t) + M_2(t)v'_0(t) \\ & \leq v''_0(t) + M_1(t)v_0(t) + M_2(t)v'_0(t). \end{aligned} \tag{41}$$

Moreover, we obtain

$$\{f(t, B\eta, F\eta, \eta, \eta', TB\eta, SB\eta) + M_1(t)\eta + M_2(t)\eta' \mid \eta \in [u_0, v_0]\}$$

is a bounded set. Hence, there exists a constant $\gamma > 0$ such that

$$\begin{aligned} & \|f(t, (Bu_{n-1})(t), (Fu_{n-1})(t), u_{n-1}(t), u'_{n-1}(t), \\ & \quad (TBu_{n-1})(t), (SBu_{n-1})(t)) - M_1(t)(u_n(t) - u_{n-1}(t)) \\ & \quad - M_2(t)(u'_n(t) - u'_{n-1}(t))\| \leq \gamma, \forall t \in J (n=1, 2, \dots), \end{aligned} \tag{42}$$

and $\{\sigma_n \mid n=1, 2, \dots\}$ is a bounded set in $PC[J, E]$, where

$$\begin{aligned} \sigma_n(t) &= f(t, (Bu_{n-1})(t), (Fu_{n-1})(t), u_{n-1}(t), u'_{n-1}(t), \\ & \quad (TBu_{n-1})(t), (SBu_{n-1})(t)) \\ & \quad + M_1(t)u_{n-1}(t) + M_2(t)u'_{n-1}(t). \end{aligned}$$

By the definition of $u_n(t)$ and (23), we have

$$u_n(t) = x_2^* + tx_3^* + \int_0^t (t-s)(f(s, (Bu_{n-1})(s),$$

$$\begin{aligned} & (Fu_{n-1})(s), u_{n-1}(s), u'_{n-1}(s), (TBu_{n-1})(s), \\ & \quad (SBu_{n-1})(s)) + M_1(s)u_{n-1}(s) \\ & \quad + M_2(s)u'_{n-1}(s) - M_1(s)u_n(s) \\ & \quad - M_2(s)u'_n(s) ds + \sum_{0 < t_k < t} (I_{2k}(u'_{n-1}(t_k)) \\ & \quad + C_k(u'_n(t_k) - u'_{n-1}(t_k))) \\ & \quad + \sum_{0 < t_k < t} (t-t_k)(I_{3k}((Bu_{n-1})(t_k), \\ & \quad (Fu_{n-1})(t_k), u_{n-1}(t_k), u'_{n-1}(t_k)) \\ & \quad - L_k(u_n(t_k) - u_{n-1}(t_k)) \\ & \quad - L_k^*(u'_n(t_k) - u'_{n-1}(t_k))), \forall t \in J (n=1, 2, \dots). \end{aligned} \tag{43}$$

Then, we have

$$\begin{aligned} u'_n(t) &= x_3^* + \int_0^t (f(s, (Bu_{n-1})(s), (Fu_{n-1})(s), \\ & \quad u_{n-1}(s), u'_{n-1}(s), (TBu_{n-1})(s), (SBu_{n-1})(s)) \\ & \quad + M_1(s)u_{n-1}(s) + M_2(s)u'_{n-1}(s) \\ & \quad - M_1(s)u_n(s) - M_2(s)u'_n(s)) ds \\ & \quad + \sum_{0 < t_k < t} (I_{3k}((Bu_{n-1})(t_k), (Fu_{n-1})(t_k), \\ & \quad u_{n-1}(t_k), u'_{n-1}(t_k)) - L_k(u_n(t_k) - u_{n-1}(t_k)) \\ & \quad - L_k^*(u'_n(t_k) - u'_{n-1}(t_k))), \forall t \in J (n=1, 2, \dots). \end{aligned} \tag{44}$$

By (42), (43) and (44), the functions belonging to U, U' are equiv-continuity on $J_k (k=0, 1, 2, \dots, m)$.

So by Lemma 2.4, we have $\forall t \in J$

$$\alpha_{PC^1}(U) = \max \left\{ \sup_{t \in J} \alpha(U(t)), \sup_{t \in J} \alpha(U'(t)) \right\}.$$

By (H_3) , there exist constants $d \geq 0, d^* \geq 0$ and $b^{(k)} \geq 0, a^{(k)} \geq 0 (k=1, 2, \dots, m)$ such that

$$\begin{aligned} & \alpha(f(t, (BU)(t), (FU)(t), U(t), U'(t), \\ & \quad (TBU)(t), (SBU)(t))) \\ & \leq d\alpha(U(t)) + d^*\alpha(U'(t)), \forall t \in J. \end{aligned} \tag{45}$$

$$\begin{aligned} & \alpha(I_{3k}((BU)(t_k), (FU)(t_k), U(t_k), U'(t_k))) \\ & \leq b^{(k)} \max \{ \alpha(U(t_k)), \alpha(U'(t_k)) \} \\ & \quad (k=1, 2, \dots, m). \end{aligned} \tag{46}$$

$$\alpha(I_{2k}(U'(t_k))) \leq a^{(k)} \alpha(U'(t_k)) (k=1, 2, \dots, m) \tag{47}$$

Hence, for any $t \in J$, combing (43), (45), (46), (47) and Lemma 2.3, we have

$$\begin{aligned} \alpha(U(t)) &\leq 2a \int_0^t (\alpha(f(s, (BU)(s), (FU)(s), \\ & \quad U(s), U'(s), (TBU)(s), (SBU)(s))) \\ & \quad + 2M_1^* \alpha(U(s)) + 2M_2^* \alpha(U'(s))) ds \end{aligned}$$

$$\begin{aligned}
 & + \sum_{0 < t_k < t} (a^{(k)} \alpha(U'(t_k)) + 2C_k \alpha(U'(t_k))) \\
 & + \sum_{0 < t_k < t} (ab^{(k)} \max\{\alpha(U(t_k)), \alpha(U'(t_k))\}) \\
 & + 2aL_k \alpha(U(t_k)) + 2aL_k^* \alpha(U'(t_k)) \\
 & \leq 2a \int_0^t (d\alpha(U(s)) + d^* \alpha(U'(s))) \\
 & + 2M_1^* \alpha(U(s)) + 2M_2^* \alpha(U'(s)) ds \\
 & + \sum_{0 < t_k < t} (a^{(k)} \alpha(U'(t_k)) + 2C_k \alpha(U'(t_k))) \\
 & + \sum_{0 < t_k < t} (ab^{(k)} \max\{\alpha(U(t_k)), \alpha(U'(t_k))\}) \\
 & + 2aL_k \alpha(U(t_k)) + 2aL_k^* \alpha(U'(t_k)) \quad (48)
 \end{aligned}$$

$$\begin{aligned}
 \alpha(U'(t)) & \leq 2 \int_0^t (d\alpha(U(s)) + d^* \alpha(U'(s))) \\
 & + 2M_1^* \alpha(U(s)) + 2M_2^* \alpha(U'(s)) ds \\
 & + \sum_{0 < t_k < t} (b^{(k)} \max\{\alpha(U(t_k)), \alpha(U'(t_k))\}) \\
 & + 2L_k \alpha(U(t_k)) + 2L_k^* \alpha(U'(t_k)) \quad (49)
 \end{aligned}$$

Let

$$m(t) = \max\{\alpha(U(t)), \alpha(U'(t))\}.$$

Because the functions belonging to U, U' are equiv-continuity on $J_k (k = 1, 2, \dots, m)$ and U, U' are bounded, we have $m(t) \in PC[J, E]$, $m(t) \geq 0$. Combing (48) and (49), we obtain

$$\begin{aligned}
 m(t) & \leq 2(a+1)(d+d^* + 2M_1^* + 2M_2^*) \int_0^t m(s) ds \quad (50) \\
 & + \sum_{0 < t_k < t} (a^{(k)} + 2C_k + (a+1)(b^{(k)} + 2L_k + 2L_k^*)) m(t_k).
 \end{aligned}$$

Moreover, by Lemma 2.2, we have $m(t) \leq 0$, this means $m(t) \equiv 0, t \in J$, moreover,

$$\alpha(U(t)) \equiv 0, \alpha(U'(t)) \equiv 0, t \in J.$$

This yields that U possesses the relatively compactness in $PC^1[J, E]$, U' possesses the relatively compactness in $PC[J, E]$. Hence, by (40) and the normality of P , $\{u_n\}$ is convergent at $u^* \in PC^1[J, E]$, $\{u'_n\}$ is convergent at $(u^*)'$ and

$$\|u_n - u^*\|_{PC^1} \rightarrow 0, \|u'_n - (u^*)'\|_{PC} \rightarrow 0 \quad (51)$$

Because f is continuous, by the definition of σ_n and (51), we have

$$\|\sigma_n - \sigma^*\|_{PC} \rightarrow 0, (n \rightarrow \infty) \quad (52)$$

where

$$\begin{aligned}
 \sigma^*(t) & = f(t, (Bu^*)(t), (Fu^*)(t), u^*(t), \\
 & (u^*)'(t), (TBu^*)(t), (SBu^*)(t))
 \end{aligned}$$

$$+ M_1(t)u^*(t) + M_2(t)(u^*)'(t).$$

By (42), (51), (52) and Lebesgue control convergent theorem, we have

$$\lim_{n \rightarrow \infty} u_n = u^*(t), \lim_{n \rightarrow \infty} u'_n = (u^*)'(t)$$

Moreover,

$$\begin{aligned}
 u^*(t) & = x_2^* + tx_3^* + \int_0^t (t-s)f(s, (Bu^*)(s), \\
 & (Fu^*)(s), u^*(s), (u^*)'(s), (TBu^*)(s), \\
 & (SBu^*)(s)) ds + \sum_{0 < t_k < t} I_{2k}((u^*)'(t_k)) \\
 & + \sum_{0 < t_k < t} (t-t_k)I_{3k}((Bu^*)(t_k), (Fu^*)(t_k), \\
 & u^*(t_k), (u^*)'(t_k)), \forall t \in J,
 \end{aligned}$$

$$\begin{aligned}
 (u^*)'(t) & = x_3^* + \int_0^t f(s, (Bu^*)(s), (Fu^*)(s), u^*(s), \\
 & (u^*)'(s), (TBu^*)(s), (SBu^*)(s)) ds \\
 & + \sum_{0 < t_k < t} I_{3k}((Bu^*)(t_k), (Fu^*)(t_k), \\
 & u^*(t_k), (u^*)'(t_k)), \forall t \in J,
 \end{aligned}$$

It is easy to prove that $u^* \in PC^1[J, E] \cap C^2[J', E]$ is a solution of IVP (18). In the same way, there exists $v^* \in PC^1[J, E] \cap C^2[J', E]$ such that

$$\|v_n - v^*\|_{PC^1} \rightarrow 0, \|v'_n - (v^*)'\|_{PC} \rightarrow 0.$$

v^* is a solution of IVP (18), and by (40), we have

$$\begin{aligned}
 u_0(t) \leq u_1(t) \leq \dots \leq u_n(t) \leq \dots \leq u^*(t) \leq v^*(t) \\
 \leq \dots \leq v_n(t) \leq \dots \leq v_1(t) \leq v_0(t), \forall t \in J. \quad (53)
 \end{aligned}$$

$$\begin{aligned}
 u'_0(t) \leq u'_1(t) \leq \dots \leq u'_n(t) \leq \dots \leq (u^*)'(t) \leq (v^*)'(t) \\
 \leq \dots \leq v'_n(t) \leq \dots \leq v'_1(t) \leq v'_0(t) \forall t \in J.
 \end{aligned}$$

For $u \in PC^1[J, E] \cap C^2[J', E]$ is any solution of IVP (18) on $[u_0, v_0]$, then

$$u_0(t) \leq u(t) \leq v_0(t), u'_0(t) \leq u'(t) \leq v'_0(t), \forall t \in J.$$

Assume that $u_{n-1}(t) \leq u(t) \leq v_{n-1}(t), u'_{n-1}(t) \leq u'(t) \leq v'_{n-1}(t), \forall t \in J$. Let $p(t) = u_n(t) - u(t)$. By (22),

(39) and (H_2) , we have

$$\begin{aligned}
 p''(t) & = -M_1(t)p(t) - M_2(t)p'(t) \\
 & - (f(t, (Bu)(t), (Fu)(t), u(t), u'(t), \\
 & (TBu)(t), (SBu)(t)) \\
 & - f(t, (Bu_{n-1})(t), (Fu_{n-1})(t), u_{n-1}(t), \\
 & u'_{n-1}(t), (TBu_{n-1})(t), (SBu_{n-1})(t)) \\
 & + M_1(t)(u(t) - u_{n-1}(t)) \\
 & + M_2(t)(u'(t) - u'_{n-1}(t)))
 \end{aligned}$$

$$\begin{aligned} &\leq -M_1(t)p(t) - M_2(t)p'(t), \forall t \in J, t \neq t_k, \\ \Delta p|_{t=t_k} &= I_{2k}(u'_{n-1}(t_k)) + C_k(u'_n(t_k) - u'_{n-1}(t_k)) \\ &\quad - I_{2k}(u'(t_k)) = C_k p'(t_k), \\ \Delta p'|_{t=t_k} &= I_{3k}((Bu_{n-1})(t_k), (Fu_{n-1})(t_k), u_{n-1}(t_k), \\ &\quad u'_{n-1}(t_k)) - L_k(u_n(t_k) - u_{n-1}(t_k)) \\ &\quad - L_k^*(u'_n(t_k) - u'_{n-1}(t_k)) \\ &\quad - I_{3k}((Bu)(t_k), (Fu)(t_k), u(t_k), u'(t_k)) \\ &\leq -L_k p(t_k) - L_k^* p'(t_k) \quad (k = 1, 2, \dots, m), \\ p'(0) &= p(0) = \theta. \end{aligned}$$

By Lemma 2.1, we have $p(t) \leq \theta, p'(t) \leq \theta, \forall t \in J$. Moreover, $u_n(t) \leq u(t), u'_n(t) \leq u'(t), \forall t \in J$. In the same way, we can show that

$$u(t) \leq v_n(t), u'(t) \leq v'_n(t), \forall t \in J,$$

Hence, we obtain

$$\begin{aligned} u_n(t) &\leq u(t) \leq v_n(t), u'_n(t) \leq u'(t) \leq v'_n(t), \\ &\quad \forall t \in J (n = 0, 1, 2, \dots). \end{aligned} \tag{54}$$

Now if $n \rightarrow \infty$, for any $t \in J$,

$$u^*(t) \leq u(t) \leq v^*(t), (u^*)'(t) \leq u'(t) \leq (v^*)'(t).$$

By (53), then (22) holds. This ends the proof.

Theorem 3.2. Suppose E is a real Banach space, P is a regular cone, and $(H_1), (H_2)$ hold. Assume (4) or (5) is satisfied, then (21) holds.

Proof. The proof is similar to the proof of Theorem 3.1, the only difference is that we verify relative compactness of U, U' and the regularity of P by (40) instead of H_3 . This ends the proof.

Corollary 3.1. If E is a weak sequentially complete Banach space, P is a normal cone, H_1, H_2 hold, and (4) or (5) is satisfied, then (21) holds.

Proof. If E is a weak sequentially complete Banach space, the normality of P is equivalent to the regularity. Hence, (21) holds by Theorem 3.2. This ends the proof.

Remark 3.1. f is relative to operators B, F . To my knowledge, in all papers connected with the second order impulsive integro-differential equation has been not investigated this situation, so IVP (18) is a new problem.

Remark 3.2. B, F relative to Theorem 3.1 are bounded and continuous operators, however, B, F relative to Theorem 3.2 are continuous and increasing.

4. Some results of the four order impulsive differential equations

Let us list the following assumptions for convenience:

(G_1) There exist

$$y_0, z_0 \in PC^3[J, E] \cap C^4[J', E]$$

such that $y_0(t) \leq z_0(t), y'_0(t) \leq z'_0(t), y''_0(t) \leq z''_0(t), y'''_0(t) \leq z'''_0(t), t \in J$,

$$\left\{ \begin{aligned} &y_0^{(4)}(t) \leq f(t, y_0(t), y'_0(t), y''_0(t), y'''_0(t), \\ &\quad (Ty_0)(t), (Sy_0)(t)), \forall t \in J, t \neq t_k, \\ \Delta y_0|_{t=t_k} &= I_{0k}(y''_0(t_k)), \\ \Delta y'_0|_{t=t_k} &= I_{1k}(y'_0(t_k), y''_0(t_k)), \\ \Delta y''_0|_{t=t_k} &= I_{2k}(y'''_0(t_k)), (k = 1, 2, \dots, m), \\ \Delta y'''_0|_{t=t_k} &\leq I_{3k}(y_0(t_k), y'_0(t_k), y''_0(t_k), y'''_0(t_k)) \\ y_0(0) &= x_0^*, y'_0(0) = x_1^*, y''_0(0) \leq x_2^*, \\ &y'''_0(0) - y''_0(0) \leq x_3^* - x_2^*. \end{aligned} \right. \tag{55}$$

$$\left\{ \begin{aligned} &z_0^{(4)}(t) \geq f(t, z_0(t), z'_0(t), z''_0(t), z'''_0(t), (Tz_0)(t), \\ &\quad (Sz_0)(t)), \forall t \in J, t \neq t_k, \\ \Delta z_0|_{t=t_k} &= I_{0k}(z''_0(t_k)), \\ \Delta z'_0|_{t=t_k} &= I_{1k}(z'_0(t_k), z''_0(t_k)), \\ \Delta z''_0|_{t=t_k} &= I_{2k}(z'''_0(t_k)), (k = 1, 2, \dots, m), \\ \Delta z'''_0|_{t=t_k} &\geq I_{3k}(z_0(t_k), z'_0(t_k), z''_0(t_k), z'''_0(t_k)), \\ z_0(0) &= x_0^*, z'_0(0) = x_1^*, z''_0(0) \geq x_2^*, \\ &z'''_0(0) - z''_0(0) \geq x_3^* - x_2^* \end{aligned} \right. \tag{56}$$

(G'_1) There exist

$$y_0, z_0 \in PC^3[J, E] \cap C^4[J', E]$$

such that $y_0(t) \leq z_0(t), y'_0(t) \leq z'_0(t), y''_0(t) \leq z''_0(t), y'''_0(t) \leq z'''_0(t), t \in J$,

$$\left\{ \begin{aligned} &y_0^{(4)}(t) \leq f(t, y_0(t), y'_0(t), y''_0(t), y'''_0(t), \\ &\quad (Ty_0)(t), (Sy_0)(t)), \forall t \in J, t \neq t_k, \\ \Delta y_0|_{t=t_k} &\leq I_{0k}(y''_0(t_k)), \\ \Delta y'_0|_{t=t_k} &\leq I_{1k}(y'_0(t_k), y''_0(t_k)), \\ \Delta y''_0|_{t=t_k} &= I_{2k}(y'''_0(t_k)), \\ \Delta y'''_0|_{t=t_k} &\leq I_{3k}(y_0(t_k), y'_0(t_k), y''_0(t_k), y'''_0(t_k)), \\ y_0(0) &\leq x_0^*, y'_0(0) \leq x_1^*, y''_0(0) \leq x_2^*, \\ &y'''_0(0) - y''_0(0) \leq x_3^* - x_2^*, \end{aligned} \right. \tag{57}$$

$$\left\{ \begin{aligned} z_0^{(4)}(t) &\geq f(t, z_0(t), z_0'(t), z_0''(t), z_0'''(t), (Tz_0)(t), \\ &\quad (Sz_0)(t)), \forall t \in J, t \neq t_k, \\ \Delta z_0|_{t=t_k} &\geq I_{0k}(z_0''(t_k)), \\ \Delta z_0'|_{t=t_k} &\geq I_{1k}(z_0'(t_k), z_0''(t_k)), \\ \Delta z_0''|_{t=t_k} &= I_{2k}(z_0'''(t_k)), \\ \Delta z_0'''|_{t=t_k} &\geq I_{3k}(z_0(t_k), z_0'(t_k), z_0''(t_k), z_0'''(t_k)), \\ &\quad (k=1, 2, \dots, m), \\ z_0(0) &\geq x_0^*, z_0'(0) \geq x_1^*, z_0''(0) \geq x_2^*, \\ z_0'''(0) - z_0''(0) &\geq x_3^* - x_2^* \end{aligned} \right. \quad (58)$$

(G₂) There exist M₁(t), M₂(t) are bounded with M₁ ≥ 0, M₂ ≥ 0 on J and M₁, M₂ ∈ L¹[0, a]. C_k, L_k, L_k^{*} (k=1, 2, …, m) are all nonnegative constants such that

$$\begin{aligned} &f(t, x, y, z, u, v, w) - f(t, \bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}) \\ &\geq -M_1(t)(z - \bar{z}) - M_2(t)(u - \bar{u}), \forall t \in J, \\ &I_{0k}(z) \geq I_{0k}(\bar{z}), I_{1k}(y, z) \geq I_{1k}(\bar{y}, \bar{z}), \\ &I_{2k}(u) - I_{2k}(\bar{u}) = C_k(u - \bar{u}) \\ &I_{3k}(x, y, z, u) - I_{3k}(\bar{x}, \bar{y}, \bar{z}, \bar{u}) \\ &\geq -L_k(z - \bar{z}) - L_k^*(u - \bar{u}) \quad (k=1, 2, \dots, m) \end{aligned}$$

where

$$\begin{aligned} y_0(t) &\leq \bar{x} \leq x \leq z_0(t), y_0'(t) \leq \bar{y} \leq y \leq z_0'(t), \\ y_0''(t) &\leq \bar{z} \leq z \leq z_0''(t), y_0'''(t) \leq \bar{u} \leq u \leq z_0'''(t), \\ (Ty_0)(t) &\leq \bar{v} \leq v \leq (Tz_0)(t), \\ (Sy_0)(t) &\leq \bar{w} \leq w \leq (Sz_0)(t), \forall t \in J \end{aligned}$$

(G₃) There exist

$$b_{0k} \geq 0, a_{1k} \geq 0, b_{1k} \geq 0, (k=1, 2, \dots, m)$$

such that

$$\begin{aligned} \|I_{0k}(z) - I_{0k}(\bar{z})\| &\leq b_{0k} \|z - \bar{z}\|, \\ \|I_{1k}(y, z) - I_{1k}(\bar{y}, \bar{z})\| &\leq a_{1k} \|y - \bar{y}\| + b_{1k} \|z - \bar{z}\|, \\ y, z, \bar{y}, \bar{z} &\in E (k=1, 2, \dots, m). \end{aligned}$$

Denote

$$\begin{aligned} [y_0, z_0] &= \{y \in PC^3[J, E] \mid y_0(t) \leq y(t) \leq z_0(t), \\ & y_0'(t) \leq y'(t) \leq z_0'(t), y_0''(t) \leq y''(t) \leq z_0''(t), \\ & y_0'''(t) \leq y'''(t) \leq z_0'''(t), \forall t \in J\}. \end{aligned}$$

Theorem 4.1 Suppose E is a real Banach space, P is normal cone, and (G₁), (G₂), (G₃), (H₃) hold. Assume (4) or (5) is satisfied, then IVP (1) has the maximal and minimal solutions

$$y^*, z^* \in PC^3[J, E] \cap C^4[J', E]$$

on [y₀, z₀].

Proof. Consider IVP (1). Let x''(t) = u(t), t ∈ J.

Then we have

$$\left\{ \begin{aligned} x''(t) &= u(t), \forall t \in J, t \neq t_k, \\ u''(t) &= f(t, x(t), x'(t), u(t), u'(t), (Tx)(t), (Sx)(t)), \\ &\quad \forall t \in J, t \neq t_k, \\ \Delta x|_{t=t_k} &= I_{0k}(u(t_k)), \\ \Delta x'|_{t=t_k} &= I_{1k}(x'(t_k), u(t_k)), \\ \Delta u|_{t=t_k} &= I_{2k}(u'(t_k)), \\ \Delta u'|_{t=t_k} &= I_{3k}(x(t_k), x'(t_k), u(t_k), u'(t_k)) \\ &\quad (k=1, 2, \dots, m), \\ x(0) &= x_0^*, x'(0) = x_1^*, u(0) = x_2^*, u'(0) = x_3^* \end{aligned} \right. \quad (59)$$

For any u ∈ PC[J, E], we have

$$\left\{ \begin{aligned} x''(t) &= u(t), \forall t \in J, t \neq t_k, \\ \Delta x|_{t=t_k} &= I_{0k}(u(t_k)), \\ \Delta x'|_{t=t_k} &= I_{1k}(x'(t_k), u(t_k)) (k=1, 2, \dots, m), \\ x(0) &= x_0^*, x'(0) = x_1^*. \end{aligned} \right. \quad (60)$$

Obviously, if x ∈ PC¹[J, E] ∩ C²[J', E] is a solution of (60) if and only if

$$\begin{aligned} x(t) &= x_0^* + x_1^*t + \int_0^t (t-s)u(s)ds + \sum_{0 < t_k < t} I_{0k}(u(t_k)) \\ &+ \sum_{0 < t_k < t} (t-t_k)I_{1k}(x'(t_k), u(t_k)), \end{aligned} \quad (61)$$

and

$$x'(t) = x_1^* + \int_0^t u(s)ds + \sum_{0 < t_k < t} I_{1k}(x'(t_k), u(t_k)). \quad (62)$$

Let

$$x(t) = (Bu)(t), t \in J. \quad (63)$$

$$x'(t) = (Fu)(t), t \in J. \quad (64)$$

Then define two operators B, F,

$$B: PC[J, E] \rightarrow PC^1[J, E] \cap C^2[J', E],$$

$$F: PC[J, E] \rightarrow PC[J, E] \cap C^1[J', E].$$

Next, we show that

(i) B is bounded and continuous.

When m = 3, 4, …, for any y₁, y₂ ∈ PC[J, E], by (62), (63), we have

$$\begin{aligned} &\| (By_1)(t) - (By_2)(t) \| \\ &\leq \int_0^t (t-s) \| y_1(s) - y_2(s) \| ds \\ &+ \sum_{0 < t_k < t} \| I_{0k}(y_1(t_k)) - I_{0k}(y_2(t_k)) \| \\ &+ \sum_{0 < t_k < t} (t-t_k) \| I_{1k}((Fy_1)(t_k), y_1(t_k)) \end{aligned}$$

$$\begin{aligned}
 & -I_{1k}((Fy_2)(t_k), y_2(t_k)) \| \\
 \leq & \frac{a^2}{2} \| y_1 - y_2 \|_{PC} + \sum_{k=1}^m b_{0k} \| y_1 - y_2 \|_{PC} \\
 & + a \sum_{k=1}^m a_{1k} \| (Fy_1)(t_k) - (Fy_2)(t_k) \| \\
 & + a \sum_{k=1}^m b_{1k} \| y_1 - y_2 \|_{PC} \\
 = & \frac{a^2}{2} \| y_1 - y_2 \|_{PC} + \sum_{k=1}^m b_{0k} \| y_1 - y_2 \|_{PC} \\
 & + a \sum_{k=1}^m b_{1k} \| y_1 - y_2 \|_{PC} \\
 & + a(a_{1m} \| (Fy_1)(t_m) - (Fy_2)(t_m) \| \\
 & + \sum_{k=1}^{m-1} a_{1k} \| (Fy_1)(t_k) - (Fy_2)(t_k) \|) \\
 \leq & \frac{a^2}{2} \| y_1 - y_2 \|_{PC} + \sum_{k=1}^m b_{0k} \| y_1 - y_2 \|_{PC} \\
 & + a \sum_{k=1}^m b_{1k} \| y_1 - y_2 \|_{PC} \\
 & + a((a_{1m} + 1) \sum_{k=1}^{m-1} a_{1k} \| (Fy_1)(t_k) - (Fy_2)(t_k) \| \\
 & + a_{1m} t_m \| y_1 - y_2 \|_{PC} + a_{1m} \sum_{k=1}^{m-1} b_{1k} \| y_1 - y_2 \|_{PC}) \\
 \leq & \frac{a^2}{2} \| y_1 - y_2 \|_{PC} + \sum_{k=1}^m b_{0k} \| y_1 - y_2 \|_{PC} \\
 & + a \sum_{k=1}^m b_{1k} \| y_1 - y_2 \|_{PC} \\
 & + a(a_{1m} \sum_{k=1}^{m-1} b_{1k} + \sum_{k=2}^{m-1} (a_{1k} (\prod_{j=k+1}^m (a_{1j} + 1)) \sum_{i=1}^{k-1} b_{1i}) \\
 & + a_{1m} t_m + \sum_{k=1}^{m-1} a_{1k} t_k \prod_{j=k+1}^m (a_{1j} + 1)) \| y_1 - y_2 \|_{PC}.
 \end{aligned} \tag{65}$$

Hence,

$$\| By_1 - By_2 \|_{PC} \leq N_1^* \| y_1 - y_2 \|_{PC},$$

where

$$\begin{aligned}
 N_1^* = & \frac{a^2}{2} + \sum_{k=1}^m b_{0k} + a \sum_{k=1}^m b_{1k} + a(a_{1m} \sum_{k=1}^{m-1} b_{1k} \\
 & + \sum_{k=2}^{m-1} (a_{1k} (\prod_{j=k+1}^m (a_{1j} + 1)) \sum_{i=1}^{k-1} b_{1i}) \\
 & + a_{1m} t_m + \sum_{k=1}^{m-1} a_{1k} t_k \prod_{j=k+1}^m (a_{1j} + 1))
 \end{aligned} \tag{67}$$

In the same way,

$$\| (By_1)' - (By_2)' \|_{PC} \leq N_2^* \| y_1 - y_2 \|_{PC},$$

where

$$\begin{aligned}
 N_2^* = & a + \sum_{k=1}^m b_{1k} + (a_{1m} \sum_{k=1}^{m-1} b_{1k} \\
 & + \sum_{k=2}^{m-1} (a_{1k} (\prod_{j=k+1}^m (a_{1j} + 1)) \sum_{i=2}^{k-1} b_{1i}) \\
 & + a_{1m} t_m + \sum_{k=1}^{m-1} a_{1k} t_k \prod_{j=k+1}^m (a_{1j} + 1)).
 \end{aligned} \tag{68}$$

So

$$\| By_1 - By_2 \|_{PC^1} \leq N^* \| y_1 - y_2 \|_{PC},$$

where $N^* = \max\{N_1^*, N_2^*\}$. Hence, B is bounded and continuous. When $m = 1, 2$, the proof is similar.

(ii) B is increasing.

For any $y_1, y_2 \in PC[J, E]$, $y_1 \leq y_2$, by (G_2) and (61), we have

$$\begin{aligned}
 & (By_1)(t) - (By_2)(t) \\
 = & \int_0^t (t-s)(y_1(s) - y_2(s)) ds \leq \theta, \quad t \in J_0.
 \end{aligned}$$

Then

$$(By_1)(t) \leq (By_2)(t), \quad \forall t \in J_0.$$

In particular, $(By_1)(t_1) \leq (By_2)(t_1)$. Moreover, for any $t \in J_1$, we have

$$\begin{aligned}
 & \sum_{0 < t_k < t} (I_{0k}(y_1(t_k)) - I_{0k}(y_2(t_k))) \\
 & + \sum_{0 < t_k < t} (t-t_k)(I_{1k}((By_1)(t_k), y_1(t_k)) \\
 & - I_{1k}((By_2)(t_k), y_2(t_k))) \\
 = & I_{01}(y_1(t_1)) - I_{01}(y_2(t_1)) \\
 & + (t-t_1)(I_{11}((By_1)(t_1), y_1(t_1)) \\
 & - I_{11}((By_2)(t_1), y_2(t_1))) \leq \theta.
 \end{aligned} \tag{69}$$

Then

$$(By_1)(t) \leq (By_2)(t), \quad \forall t \in J_1.$$

In particular, $(By_1)(t_2) \leq (By_2)(t_2)$. In the same way, we have

$$\begin{aligned}
 & (By_1)(t) \leq (By_2)(t), \quad \forall t \in J_k, \\
 & (By_1)(t_{k+1}) \leq (By_2)(t_{k+1}) \quad (k = 1, 2, \dots, m).
 \end{aligned}$$

Hence,

$$(By_1)(t) \leq (By_2)(t), \quad \forall t \in J,$$

then $By_1 \leq By_2$. In the same way, F is a bounded continuous operator with increasing. Combing (61) and (62), it is easy to show if

$$y \in PC^1[J, E] \cap C^2[J', E],$$

then $By \in PC^3[J, E] \cap C^4[J', E]$,
and if

$$y \in PC^1[J, E] \cap C^2[J', E],$$

then $Fy \in PC^2[J, E] \cap C^3[J', E]$.

In the same way, we can show

$$B:PC^1[J, E] \cap C^2[J', E] \rightarrow PC^3[J, E] \cap C^4[J', E]$$

$$F:PC^1[J, E] \cap C^2[J', E] \rightarrow PC^2[J, E] \cap C^3[J', E]$$

They are all bounded continuous operators with increasing. Hence, by (59)-(64), IVP (1) is equivalent to IVP (18).

Obviously, if $u \in PC^1[J, E] \cap C^2[J', E]$ is a solution of IVP (18), then

$$x(t) \in PC^3[J, E] \cap C^4[J', E]$$

is a solution of IVP (1) by(63).

Putting $u_0(t) = y_0''(t), v_0(t) = z_0''(t), t \in J$, we have $u_0 \leq v_0$. By (G_1) , we obtain

$$\begin{aligned} y_0(t) &= x_0^* + x_1^*t + \int_0^t (t-s)u_0(s)ds \\ &+ \sum_{0 < t_k < t} I_{0k}(u_0(t_k)) \\ &+ \sum_{0 < t_k < t} (t-t_k)I_{1k}(y_0'(t_k), u_0(t_k)), \forall t \in J, \end{aligned} \tag{70}$$

$$\begin{aligned} z_0(t) &= x_0^* + x_1^*t + \int_0^t (t-s)v_0(s)ds \\ &+ \sum_{0 < t_k < t} I_{0k}(v_0(t_k)) \\ &+ \sum_{0 < t_k < t} (t-t_k)I_{1k}(z_0'(t_k), v_0(t_k)), \forall t \in J, \end{aligned} \tag{71}$$

$$\begin{aligned} y_0'(t) &= x_1^* + \int_0^t u_0(s)ds \\ &+ \sum_{0 < t_k < t} I_{1k}(y_0'(t_k), u_0(t_k)), \forall t \in J, \end{aligned} \tag{72}$$

$$\begin{aligned} z_0'(t) &= x_1^* + \int_0^t v_0(s)ds \\ &+ \sum_{0 < t_k < t} I_{1k}(z_0'(t_k), v_0(t_k)), \forall t \in J, \end{aligned} \tag{73}$$

then

$$\begin{aligned} y_0(t) &= (Bu_0)(t), z_0(t) = (Bv_0)(t), \\ y_0'(t) &= (Fu_0)(t), z_0'(t) = (Fv_0)(t), \forall t \in J, \end{aligned}$$

where u_0, v_0 satisfy (H_1) .

By (G_2) , it is easy to know that (H_2) holds. Hence, applying Theorem3.1, there exist the maximal and minimal solutions $u^*, v^* \in PC^1[J, E] \cap C^2[J', E]$ of IVP (18) on $[u_0, v_0]$.

Let $y^* = Bu^*, z^* = Bv^*$. Then

$$y^*, z^* \in PC^3[J, E] \cap C^4[J', E]$$

and

$$\begin{aligned} y^*(t) &= x_0^* + x_1^*t + \int_0^t (t-s)u^*(s)ds \\ &+ \sum_{0 < t_k < t} I_{0k}(u^*(t_k)) \\ &+ \sum_{0 < t_k < t} (t-t_k)I_{1k}((y^*)'(t_k), u^*(t_k)), \forall t \in J. \end{aligned} \tag{74}$$

By (74), we have

$$\begin{cases} (y^*)''(t) = u^*(t), \forall t \in J, t \neq t_k, \\ \Delta y^*|_{t=t_k} = I_{0k}(u^*(t_k)), \\ \Delta (y^*)'|_{t=t_k} = I_{1k}((y^*)'(t_k), u^*(t_k)), (k = 1, 2, \dots, m), \\ y^*(0) = x_0^*, (y^*)'(0) = x_1^*, \end{cases} \tag{75}$$

If there exist u^* such that (18) and y^* such that (75), then y^* is a solution of IVP (1).In the same way, z^* is a solution of IVP (1).It is easy to verify $y^*, z^* \in PC^3[J, E] \cap C^4[J', E]$ are the maximal and minimal solutions of IVP of (1) on $[y_0, z_0]$, respectively. This ends the proof.

Theorem 4.2. Suppose E is a real Banach space, P is a regular cone, and $(G_1), (G_2), (G_3)$ hold. Assume (4) or (5) is satisfied, then there exist the maximal and minimal solutions $y^*, z^* \in PC^3[J, E] \cap C^4[J', E]$ of IVP (1) on $[y_0, z_0]$, respectively.

Proof. The proof is similar to the proof of Theorem 4.1.If Theorem 3.2 satisfies, then there exist $u^*, v^* \in PC^1[J, E] \cap C^2[J', E]$ the maximal and minimal solutions of IVP (1) respectively. This ends the proof.

Corollary 4.1. If E is a weak sequentially complete Banach space, P is a normal cone, $(G_1), (G_2), (G_3)$ hold, and (4) or (5) is satisfied, then IVP (1) has the maximal and minimal solutions $y^*, z^* \in PC^3[J, E] \cap C^4[J', E]$ on $[y_0, z_0]$.

Proof. If E is a weak sequentially complete Banach space, the normality of P is equivalent to the regularity of P .Hence the conclusion of Corollary4.1 holds by Theorem 4.2.This ends the proof.

Theorem 4.3. Suppose E is a real Banach space. P is regular cone, and $(G_1'), (G_2), (G_3), (H_3)$ hold. Assume (4) or (5) is satisfied. If for any $z, u \in E$,

$$\begin{aligned} f(t, x, y, z, u, v, w) &\geq f(t, \bar{x}, \bar{y}, z, u, \bar{v}, \bar{w}) \\ \forall x \geq \bar{x}, y &\geq \bar{y}, v \geq \bar{v}, w \geq \bar{w}, \end{aligned} \tag{76}$$

then IVP (1) has the maximal and minimal solutions

$y^*, z^* \in PC^3[J, E] \cap C^4[J', E]$ on $[y_0, z_0]$.

Proof. Similar to the proof of Theorem 4.1, we consider IVP (1). Let $x''(t) = u(t), t \in J$, then

$$x(t) = (Bu)(t), x'(t) = (Fu)(t), t \in J.$$

Hence, IVP (1) is equivalent to IVP (18). Let

$$u_0(t) = y_0''(t), v_0(t) = z_0''(t), t \in J. \quad (77)$$

Then $u_0 \leq v_0$. Combining (77) and (G'_1) , for any $t \in J$, we have

$$\begin{aligned} y_0(t) &= y_0(0) + y_0'(0)t + \int_0^t (t-s)u_0(s)ds \\ &\quad + \sum_{0 < t_k < t} \Delta y_0(t_k) + \sum_{0 < t_k < t} (t-t_k)\Delta y_0'(t_k). \\ y_0'(t) &= y_0'(0) + \int_0^t u_0(s)ds + \sum_{0 < t_k < t} \Delta y_0'(t_k), \forall t \in J, \\ z_0(t) &= z_0(0) + z_0'(0)t + \int_0^t (t-s)v_0(s)ds \\ &\quad + \sum_{0 < t_k < t} \Delta z_0(t_k) + \sum_{0 < t_k < t} (t-t_k)\Delta z_0'(t_k), \\ z_0'(t) &= z_0'(0) + \int_0^t v_0(s)ds + \sum_{0 < t_k < t} \Delta z_0'(t_k). \end{aligned}$$

It is easy to verify

$$\begin{aligned} y_0(t) &\leq (Bu_0)(t), y_0'(t) \leq (Fu_0)(t), \\ (Bv_0)(t) &\leq z_0(t), (Fv_0)(t) \leq z_0'(t), \quad t \in J_0. \end{aligned}$$

In particular,

$$\begin{aligned} y_0(t_1) &\leq (Bu_0)(t_1), y_0'(t_1) \leq (Fu_0)(t_1), \\ (Bv_0)(t_1) &\leq z_0(t_1), (Fv_0)(t_1) \leq z_0'(t_1). \end{aligned}$$

Moreover, we have for any $k, (k = 1, 2, \dots, m)$

$$\begin{aligned} y_0(t) &\leq (Bu_0)(t), y_0'(t) \leq (Fu_0)(t), \\ (Bv_0)(t) &\leq z_0(t), (Fv_0)(t) \leq z_0'(t), t \in J_k, \\ y_0(t_{k+1}) &\leq (Bu_0)(t_{k+1}), y_0'(t_{k+1}) \leq (Fu_0)(t_{k+1}), \\ (Bv_0)(t_{k+1}) &\leq z_0(t_{k+1}), (Fv_0)(t_{k+1}) \leq z_0'(t_{k+1}). \end{aligned}$$

So we have

$$y_0 \leq Bu_0, y_0' \leq Fu_0, Bv_0 \leq z_0, Fv_0 \leq z_0'.$$

Hence, by (G'_1) , we know (H_1) holds. Similar to the proof Theorem 4.1, we obtain the conclusion. This ends the proof.

Theorem 4.4. Suppose E is a real Banach space. P is regular cone, $(G'_1), (G_2), (G_3)$ hold. Assume (4) or (5) is satisfied. If for any $z, u \in E$, (76) holds, then IVP (1) has the maximal and minimal solutions

$$y^*, z^* \in PC^3[J, E] \cap C^4[J', E]$$

on $[y_0, z_0]$.

Proof. Similar to Theorem 4.3, it is easy to know (H_1) holds. Then the rest of the proof is similar to

the proof of Theorem 4.3. This ends of the proof.

Corollary 4.2. If E is a weak sequentially complete Banach space, P is a normal cone, $(G'_1), (G_2), (G_3)$ hold. Assume (4) or (5) is satisfied. If for any $z, u \in E$, (76) holds, then IVP (1) has the maximal and minimal solutions

$$y^*, z^* \in PC^3[J, E] \cap C^4[J', E]$$

on $[y_0, z_0]$.

Proof. If E is a weak sequentially complete Banach space, the normality of P is equivalent to the regularity of P . Hence, the conclusion of Corollary 4.2 holds by Theorem 4.4. This ends the proof.

Remark 4.1. In Theorem 3.2 and Theorem 4.2, Theorem 4.4, the condition (H_3) is more easy to use and verify.

5 Application

Example 5.1. Consider the following initial value problem for fourth-order impulsive integro-differential equations:

$$\left\{ \begin{aligned} x_n^{(4)}(t) &= \frac{1}{3n}(t^2 + x_{2n}(t)) + \frac{1}{4n} \left(\frac{t}{n} + x_n'(t) \right) \\ &\quad + \frac{t}{18n} \left(\frac{t^2}{2n} - x_n''(t) \right) + \frac{t^2}{9} \left(\frac{t}{n} - x_n'''(t) \right) \\ &\quad + \frac{1}{6n} \left(t + \int_0^t e^{-ts} x_n(s) ds \right) \\ &\quad + \frac{1}{2n} \int_0^1 \frac{1}{1+t+s} x_{2n}(s) ds, \forall 0 \leq t \leq 1, t \neq \frac{1}{2}, \\ \Delta x_n \Big|_{t=\frac{1}{2}} &= \frac{1}{25(n+1)^2} x_n'' \left(\frac{1}{2} \right), \\ \Delta x_n' \Big|_{t=\frac{1}{2}} &= \frac{1}{12} x_n' \left(\frac{1}{2} \right) + \frac{1}{10n^2} x_n'' \left(\frac{1}{2} \right) \\ \Delta x_n'' \Big|_{t=\frac{1}{2}} &= \frac{1}{4} x_n''' \left(\frac{1}{2} \right), \\ \Delta x_n''' \Big|_{t=\frac{1}{2}} &= \frac{1}{2n} x_n \left(\frac{1}{2} \right) + \frac{1}{4n} x_n' \left(\frac{1}{2} \right) - \frac{1}{15} x_n'' \left(\frac{1}{2} \right) \\ &\quad - \frac{1}{8n} x_n''' \left(\frac{1}{2} \right), (n = 1, 2, 3, \dots) \\ x_n(0) &= 0, x_n'(0) = 0, x_n''(0) = 0, x_n'''(0) = 0 \end{aligned} \right. \quad (78)$$

Conclusion IVP (78) has the maximal and minimal solutions belonging to C^4 on $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ such that

$$0 \leq x_n(t) \leq \begin{cases} \frac{t^4}{24n}, t \in [0, \frac{1}{2}], \\ \frac{t^4}{12n}, t \in (\frac{1}{2}, 1], \end{cases} \quad (n = 1, 2, \dots)$$

$$0 \leq x'_n(t) \leq \begin{cases} \frac{t^3}{6n}, t \in [0, \frac{1}{2}], \\ \frac{t^3}{3n}, t \in (\frac{1}{2}, 1], \end{cases} \quad (n = 1, 2, \dots)$$

$$0 \leq x''_n(t) \leq \begin{cases} \frac{t^2}{2n}, t \in [0, \frac{1}{2}], \\ \frac{t^2}{n}, t \in (\frac{1}{2}, 1], \end{cases} \quad (n = 1, 2, \dots)$$

$$0 \leq x'''_n(t) \leq \begin{cases} \frac{t}{n}, t \in [0, \frac{1}{2}], \\ \frac{2t}{n}, t \in (\frac{1}{2}, 1], \end{cases} \quad (n = 1, 2, \dots)$$

Proof. Let

$$E = c_0 = \{x = (x_1, x_2, \dots, x_n, \dots): x_n \rightarrow 0\}$$

with the norm $\|x\| = \sup_n |x_n|$,

$$P = \{x = (x_1, x_2, \dots, x_n, \dots) \in c_0: x_n \geq 0, n = 1, 2, 3, \dots\}$$

Then P is a normal cone in E , and (78) is a initial value problem in E , where

$$a = 1, k(t, s) = e^{-ts}, h(t, s) = \frac{1}{1+t+s},$$

$$x_0^* = x_1^* = x_2^* = x_3^* = (0, \dots, 0, \dots),$$

$$x = (x_1, x_2, \dots, x_n, \dots), y = (y_1, y_2, \dots, y_n, \dots),$$

$$z = (z_1, z_2, \dots, z_n, \dots), u = (u_1, u_2, \dots, u_n, \dots),$$

$$v = (v_1, v_2, \dots, v_n, \dots), w = (w_1, w_2, \dots, w_n, \dots),$$

$$f = (f_1, f_2, \dots, f_n, \dots)$$

and

$$f_n(t, x, y, z, u, v, w) = \frac{1}{3n}(t^2 + x_{2n}) + \frac{1}{4n}(\frac{t}{n} + y_n) + \frac{t}{18n}(\frac{t^2}{2n} - z_n) + \frac{t^2}{9}(\frac{t}{n} - u_n) + \frac{1}{6n}(t + v_n) + \frac{1}{2n}w_{2n}, \quad (79)$$

$$m = 1, t_1 = \frac{1}{2}, I_{01} = (I_{011}, I_{012}, \dots, I_{01n}, \dots),$$

$$I_{11} = (I_{111}, I_{112}, \dots, I_{11n}, \dots), I_{21} = (I_{211}, I_{212}, \dots, I_{21n}, \dots),$$

$$I_{31} = (I_{311}, I_{312}, \dots, I_{31n}, \dots),$$

where

$$I_{01n}(z) = \frac{1}{25(n+1)^2} z_n,$$

$$I_{11n}(y, z) = \frac{1}{12} y_n + \frac{1}{10n^2} z_n, I_{21n}(u) = \frac{1}{4} u_n,$$

$$I_{31n}(x, y, z, u) = \frac{1}{2n} x_n + \frac{1}{4n} y_n - \frac{1}{15} z_n - \frac{1}{8n} u_n.$$

Let $J = [0, 1]$, obviously,

$$f \in C[J \times E \times E \times E \times E \times E \times E, E].$$

Let $y_0(t) = (0, 0, \dots, 0, \dots), t \in [0, 1]$,

$$z_0(t) = \begin{cases} (\frac{t^4}{24}, \frac{t^4}{48}, \dots, \frac{t^4}{24n}, \dots), t \in [0, \frac{1}{2}], \\ (\frac{t^4}{12}, \frac{t^4}{24}, \dots, \frac{t^4}{12n}, \dots), t \in (\frac{1}{2}, 1]. \end{cases}$$

We have

$$y'_0(t) = (0, 0, \dots, 0, \dots), t \in [0, 1],$$

$$y''_0(t) = (0, 0, \dots, 0, \dots), t \in [0, 1],$$

$$y'''_0(t) = (0, 0, \dots, 0, \dots), t \in [0, 1],$$

$$y_0^{(4)}(t) = (0, 0, \dots, 0, \dots), t \in [0, 1],$$

$$z'_0(t) = \begin{cases} (\frac{t^3}{6}, \frac{t^3}{12}, \dots, \frac{t^3}{6n}, \dots), t \in [0, \frac{1}{2}], \\ (\frac{t^3}{3}, \frac{t^3}{6}, \dots, \frac{t^3}{3n}, \dots), t \in (\frac{1}{2}, 1], \end{cases}$$

$$z''_0(t) = \begin{cases} (\frac{t^2}{2}, \frac{t^2}{4}, \dots, \frac{t^2}{2n}, \dots), t \in [0, \frac{1}{2}], \\ (t^2, \frac{t^2}{2}, \dots, \frac{t^2}{n}, \dots), t \in (\frac{1}{2}, 1], \end{cases}$$

$$z'''_0(t) = \begin{cases} (t, \frac{t}{2}, \dots, \frac{t}{n}, \dots), t \in [0, \frac{1}{2}], \\ (2t, t, \dots, \frac{2t}{n}, \dots), t \in (\frac{1}{2}, 1], \end{cases}$$

$$z_0^{(4)}(t) = \begin{cases} (1, \frac{1}{2}, \dots, \frac{1}{n}, \dots), t \in [0, \frac{1}{2}], \\ (2, 1, \dots, \frac{2}{n}, \dots), t \in (\frac{1}{2}, 1]. \end{cases}$$

Hence, we have $y_0, z_0 \in PC^3[J, E] \cap C^4[J', E]$,

$$y'_0(t) \leq z'_0(t), y''_0(t) \leq z''_0(t), y'''_0(t) \leq z'''_0(t), t \in J$$

and

$$y_0(0) = z_0(0) = (0, 0, \dots, 0, \dots) = x_0^*,$$

$$y'_0(0) = z'_0(0) = (0, 0, \dots, 0, \dots) = x_1^*,$$

$$y''_0(0) = z''_0(0) = (0, 0, \dots, 0, \dots) = x_2^*,$$

$$y'''_0(0) = z'''_0(0) = (0, 0, \dots, 0, \dots) = x_3^*,$$

$$f_n(t, y_0(t), y_0'(t), y_0''(t), y_0'''(t), (Ty_0)(t), (Sy_0)(t)) \\ = \frac{t^2}{3n} + \frac{t}{4n^2} + \frac{t^3}{36n^2} + \frac{t^3}{9n} + \frac{t}{6n} \geq 0, \forall t \in [0, 1]$$

when $0 \leq t \leq \frac{1}{2}$,

$$f_n(t, z_0(t), z_0'(t), z_0''(t), z_0'''(t), (Tz_0)(t), (Sz_0)(t)) \\ \leq \frac{1}{3n}(t^2 + \frac{t^4}{48n}) + \frac{1}{4n}(\frac{t}{n} + \frac{t^3}{6n}) \\ + \frac{1}{6n}(t + \int_0^t \frac{s^4}{24n} ds) + \frac{1}{2n} \int_0^1 \frac{s^4}{48n} ds \leq \frac{1}{n}.$$

when $\frac{1}{2} < t \leq 1$,

$$f_n(t, z_0(t), z_0'(t), z_0''(t), z_0'''(t), (Tz_0)(t), (Sz_0)(t)) \\ \leq \frac{1}{3n}(t^2 + \frac{t^4}{24n}) + \frac{1}{4n}(\frac{t}{n} + \frac{t^3}{3n}) + \frac{t}{18n}(\frac{t^2}{2n} - \frac{t^2}{n}) \\ + \frac{t^2}{9}(\frac{t}{n} - \frac{2t}{n}) + \frac{1}{6n}(t + \int_0^t \frac{s^4}{12n} ds) + \frac{1}{2n} \int_0^1 \frac{s^4}{24n} ds \\ \leq \frac{2}{n},$$

$$\Delta y_0|_{t=\frac{1}{2}} = (0, 0, \dots, 0, \dots) = I_{01}(y_0''(\frac{1}{2})),$$

$$\Delta y_0'|_{t=\frac{1}{2}} = (0, 0, \dots, 0, \dots) = I_{11}(y_0'(\frac{1}{2}), y_0''(\frac{1}{2})),$$

$$\Delta y_0''|_{t=\frac{1}{2}} = (0, 0, \dots, 0, \dots) = I_{21}(y_0'''(\frac{1}{2})),$$

$$\Delta y_0'''|_{t=\frac{1}{2}} = (0, 0, \dots, 0, \dots) \\ = I_{31}(y_0(\frac{1}{2}), y_0'(\frac{1}{2}), y_0''(\frac{1}{2}), y_0'''(\frac{1}{2})),$$

$$\Delta z_0|_{t=\frac{1}{2}} = (\frac{1}{384}, \frac{1}{768}, \dots, \frac{1}{384n}, \dots) \geq I_{01}(z_0''(\frac{1}{2})),$$

$$\Delta z_0'|_{t=\frac{1}{2}} = (\frac{1}{48}, \frac{1}{96}, \dots, \frac{1}{48n}, \dots) \geq I_{11}(z_0'(\frac{1}{2}), z_0''(\frac{1}{2})),$$

$$\Delta z_0''|_{t=\frac{1}{2}} = (\frac{1}{8}, \frac{1}{16}, \dots, \frac{1}{8n}, \dots) = I_{21}(z_0'''(\frac{1}{2})),$$

$$\Delta z_0'''|_{t=\frac{1}{2}} = (\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2n}, \dots) \\ \geq I_{31}(z_0(\frac{1}{2}), z_0'(\frac{1}{2}), z_0''(\frac{1}{2}), z_0'''(\frac{1}{2})),$$

so (G_1') is satisfied. On the other hand, for any $t \in J$,

$$y_0(t) \leq \bar{x} \leq x \leq z_0(t), y_0'(t) \leq \bar{y} \leq y \leq z_0'(t),$$

$$y_0''(t) \leq \bar{z} \leq z \leq z_0''(t), y_0'''(t) \leq \bar{u} \leq u \leq z_0'''(t), \\ Ty_0(t) \leq \bar{v} \leq v \leq Tz_0(t), Sy_0(t) \leq \bar{w} \leq w \leq Sz_0(t)$$

we have

$$f(t, x, y, z, u, v, w) - f(t, \bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}, \bar{w}) \\ = \frac{1}{3n}(x_{2n} - \bar{x}_{2n}) + \frac{1}{4n}(y_n - \bar{y}_n) + \frac{t}{18n}(\bar{z}_n - z_n) \\ + \frac{t^2}{9}(\bar{u}_n - u_n) + \frac{1}{6n}(v_n - \bar{v}_n) + \frac{1}{2n}(w_{2n} - \bar{w}_{2n}) \\ \geq -\frac{t}{18}(z_n - \bar{z}_n) - \frac{t^2}{9}(u_n - \bar{u}_n) (n=1, 2, 3, \dots)$$

$$I_{01}(z) \geq I_{01}(\bar{z}), I_{11}(y, z) \geq I_{11}(\bar{y}, \bar{z}),$$

$$I_{21}(u) - I_{21}(\bar{u}) = \frac{1}{4}(u - \bar{u}),$$

$$I_{31}(x, y, z, u) - I_{31}(\bar{x}, \bar{y}, \bar{z}, \bar{u}) \\ \geq -\frac{1}{15}(z - \bar{z}) - \frac{1}{8}(u - \bar{u}),$$

so (G_2) is satisfied, where

$$M_1(t) = \frac{t}{18}, M_2(t) = \frac{t^2}{9}, C_1 = \frac{1}{4}, L_1 = \frac{1}{15}, L_1^* = \frac{1}{8},$$

It is easy to verify (4) holds.

Obviously, for any $y, z, \bar{y}, \bar{z} \in E$, we have

$$\|I_{01}(z) - I_{01}(\bar{z})\| \leq \frac{1}{100} \|z - \bar{z}\|,$$

$$\|I_{11}(y, z) - I_{11}(\bar{y}, \bar{z})\| \leq \frac{1}{12} \|y - \bar{y}\| + \frac{1}{10} \|z - \bar{z}\|,$$

so (G_3) is satisfied. By (79), then

$$f = f^{(1)} + f^{(2)}, f^{(1)} = (f_1^{(1)}, f_2^{(1)}, \dots, f_n^{(1)}, \dots), \\ f^{(2)} = (f_1^{(2)}, f_2^{(2)}, \dots, f_n^{(2)}, \dots),$$

where

$$f_n^{(1)}(t, x, y, z, u, v, w) \\ = \frac{1}{3n}(t^2 + x_{2n}) + \frac{1}{4n}(\frac{t}{n} + y_n) \\ + \frac{t}{18n}(\frac{t^2}{2n} - z_n) + \frac{1}{6n}(t + v_n) + \frac{1}{2n}w_{2n}, \quad (80)$$

$$f_n^{(2)}(t, x, y, z, u, v, w) = \frac{t^2}{9}(\frac{t}{n} - u_n). \quad (81)$$

For any $r > 0$, assume $\{t^{(b)}\}_{b=1}^\infty \subset J$,

$$\{x^{(b)}\}_{b=1}^\infty, \{y^{(b)}\}_{b=1}^\infty, \{z^{(b)}\}_{b=1}^\infty, \{u^{(b)}\}_{b=1}^\infty, \{v^{(b)}\}_{b=1}^\infty, \\ \{w^{(b)}\}_{b=1}^\infty \subset B_r,$$

where

$$x^{(b)} = (x_1^{(b)}, x_2^{(b)}, \dots, x_n^{(b)}, \dots),$$

$$y^{(b)} = (y_1^{(b)}, y_2^{(b)}, \dots, y_n^{(b)}, \dots),$$

$$\begin{aligned} z^{(b)} &= (z_1^{(b)}, z_2^{(b)}, \dots, z_n^{(b)}, \dots), \\ u^{(b)} &= (u_1^{(b)}, u_2^{(b)}, \dots, u_n^{(b)}, \dots), \\ v^{(b)} &= (v_1^{(b)}, v_2^{(b)}, \dots, v_n^{(b)}, \dots), \\ w^{(b)} &= (w_1^{(b)}, w_2^{(b)}, \dots, w_n^{(b)}, \dots) \end{aligned}$$

By (80), we have

$$\begin{aligned} &f_n^{(1)}(t^{(b)}, x^{(b)}, y^{(b)}, z^{(b)}, u^{(b)}, v^{(b)}, w^{(b)}) \\ &\leq \frac{1}{3n}(1 + \|x^{(b)}\|) + \frac{1}{4n}(\frac{1}{n} + \|y^{(b)}\|) \\ &+ \frac{1}{18n}(\frac{1}{2n} + \|z^{(b)}\|) + \frac{1}{6n}(1 + \|v^{(b)}\|) + \frac{1}{2n}\|w^{(b)}\| \\ &\leq \frac{1}{3n}(1+r) + \frac{1}{4n}(\frac{1}{n}+r) + \frac{1}{18n}(\frac{1}{2n}+r) \\ &+ \frac{1}{6n}(1+r) + \frac{1}{2n}r(b, n=1, 2, 3, \dots). \end{aligned} \tag{82}$$

So

$$\{f_n^{(1)}(t^{(b)}, x^{(b)}, y^{(b)}, z^{(b)}, u^{(b)}, v^{(b)}, w^{(b)})\}$$

is bounded, moreover, we choose subsequence $\{b_i\} \subset \{b\}$ such that

$$\begin{aligned} f_n^{(1)}(t^{(b_i)}, x^{(b_i)}, y^{(b_i)}, z^{(b_i)}, u^{(b_i)}, v^{(b_i)}, w^{(b_i)}) &\rightarrow \zeta_n, \\ i &\rightarrow \infty (n=1, 2, 3, \dots). \end{aligned} \tag{83}$$

Combing (82) and (83), we have

$$\begin{aligned} |\zeta_n| &\leq \frac{1}{3n}(1+r) + \frac{1}{4n}(\frac{1}{n}+r) + \frac{1}{18n}(\frac{1}{2n}+r) \\ &+ \frac{1}{6n}(1+r) + \frac{1}{2n}r(n=1, 2, 3, \dots), \end{aligned} \tag{84}$$

so $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n, \dots) \in c_0 = E$. For any $\varepsilon > 0$, by (82) and (84), there exists a positive integer n_0 such that

$$\begin{aligned} |f_n^{(1)}(t^{(b_i)}, x^{(b_i)}, y^{(b_i)}, z^{(b_i)}, u^{(b_i)}, v^{(b_i)}, w^{(b_i)})| &< \varepsilon, \\ |\zeta_n| &< \varepsilon, \forall n > n_0, (i=1, 2, 3, \dots). \end{aligned} \tag{85}$$

By (83), there exists a positive integer i_0 such that

$$\begin{aligned} |f_n^{(1)}(t^{(b_i)}, x^{(b_i)}, y^{(b_i)}, z^{(b_i)}, u^{(b_i)}, v^{(b_i)}, w^{(b_i)}) - \zeta_n| &< \varepsilon, \\ \forall i > i_0, (n=1, 2, \dots, n_0). \end{aligned} \tag{86}$$

Then, combining (85) and (86), we have

$$\begin{aligned} &\|f^{(1)}(t^{(b_i)}, x^{(b_i)}, y^{(b_i)}, z^{(b_i)}, u^{(b_i)}, v^{(b_i)}, w^{(b_i)}) - \zeta\| \\ &= \sup_n |f_n^{(1)}(t^{(b_i)}, x^{(b_i)}, y^{(b_i)}, z^{(b_i)}, u^{(b_i)}, v^{(b_i)}, w^{(b_i)}) - \zeta_n| \\ &\leq 2\varepsilon, \forall i > i_0. \end{aligned}$$

Hence,

$$\begin{aligned} &\|f^{(1)}(t^{(b_i)}, x^{(b_i)}, y^{(b_i)}, z^{(b_i)}, u^{(b_i)}, v^{(b_i)}, w^{(b_i)}) - \zeta\| \\ &\rightarrow 0, \quad i \rightarrow \infty. \end{aligned}$$

Thus,

$$\alpha(f^{(1)}(J, U_1, U_2, U_3, U_4, U_5, U_6)) = 0,$$

$$\forall U_i \subset B_r (i=1, 2, 3, 4, 5, 6). \tag{87}$$

On the other hand, applying (81),

$$\begin{aligned} \alpha(f^{(2)}(J, U_1, U_2, U_3, U_4, U_5, U_6)) &\leq \frac{1}{9}\alpha(U_4), \\ \forall U_i \subset B_r (i=1, 2, 3, 4, 5, 6) \end{aligned} \tag{88}$$

By (87) and (88), we have

$$\begin{aligned} \alpha(f(J, U_1, U_2, U_3, U_4, U_5, U_6)) &\leq \frac{1}{9}\alpha(U_4), \\ \forall U_i \subset B_r (i=1, 2, 3, 4, 5, 6). \end{aligned} \tag{89}$$

In the same way,

$$\begin{aligned} \alpha(I_{31}(V_1, V_2, V_3, V_4)) &\leq \frac{1}{15}\alpha(V_3), \\ \forall V_j \subset B_r (j=1, 2, 3, 4), \end{aligned} \tag{90}$$

$$\alpha(I_{21}(V_4)) \leq \frac{1}{4}\alpha(V_4), \forall V_4 \subset B_r. \tag{91}$$

Hence, (H_3) holds, where

$$d_r = 0, d_r^* = \frac{1}{9}, b_r^{(1)} = \frac{1}{15}, a_r^{(1)} = \frac{1}{4}.$$

Finally, it is easy to prove (76) holds. Then, we have the conclusion by Theorem 4.3. This ends the proof.

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