Existence and iterative algorithm of solutions for a new system of generalized set-valued mixed equilibrium-like problems in Banach spaces

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Abstract: A new system of generalized set-valued mixed equilibrium-like problems (in short, S-GMELP) in Banach spaces is discussed. In order to obtain the existence of solutions of S-GMELP, a system of related auxiliary problems (in short, S-AP) is established. On the basis of the existence and uniqueness of solutions of the S-AP, an iterative algorithm for the S-GMELP is constructed. It is proved that the iterative sequence converges some solution of S-GMELP. Finally, an example is given to well exemplify our main result.

Key–Words: Equilibrium-like problem, Auxiliary principle technique, Existence, Iterative algorithm, Banach space

1 Introduction

When Hartman-Stampacchia [1] created the variational inequality theory in 1966, they researched the first variational inequality defined on finite-dimension spaces. In recent years, many authors have generalized the variational inequality problems in two main aspects: the model of the variational inequalities and the framework of spaces mainly including Hilbert spaces and Banach spaces. Parida-Sen [2] in 1987 introduced the variational-like inequality problems closely related to the convex mathematical programming problems in finite-dimension spaces. In [3-10] authors discussed the mixed variational-like inequality problems (in short, MVLIP) in Banach spaces. The authors in [11-14] generalized the MVLIP to the generalized set-valued mixed variational-like inequality problems (in short, GMVLIP), and investigated the GMVLIP in Hilbert spaces, and so did the authors in [15, 16] in Banach spaces.

In the sequel, Kazmi-Khan [17] presented a system of MVLIP (in short, S-MVLIP), constructed an algorithm for it by exploiting auxiliary principle technique and investigated the convergence analysis of the algorithm in Hilbert spaces. Recently, Wang-Ding [18] generalized the S-MVLIP to a system of GMVLIP (in short, S-GMVLIP) and studied the S-GMVLIP in Banach spaces.

The equilibrium theory is one of the most important tools to analyze a system, since it provides a new unified framework for those problems which arise from physics, economics, traffic, optimization theory, etc. As a matter of fact, equilibrium problems include variational inequality problems, fixed-point problems, mathematical programming problems and others as their special cases. The class of equilibrium-like problems is a useful generalization of the class of variational-like inequality problems and has some potential and significant applications in optimization and economics. To the best of our knowledge, the results on the existence of solutions of equilibrium-like problems by constructing iterative algorithms are few. Inspired by the recent works involving the system of variational-like inequality problems in [17, 18], we introduce a system of generalized set-valued mixed equilibrium-like problems and discussed the existence of its solutions by using auxiliary principle technique in Banach spaces. The introduced system includes the problems discussed in [3-17] as its special cases and its framework of spaces is generalized from Hilbert spaces in [5, 6, 11-15, 17] to Banach spaces. The rest of this paper is organized as follows. In Section 2, the main problem discussed in this paper is presented and some preliminaries are list. In Sections 3, a theorem on the existence and uniqueness of solutions of a system of related auxiliary problems is proved. In Section 4, by the theorem presented in Section 3, an iterative algorithm for the main problem is proposed. In Section 5, a main theorem on the existence of solutions of the main problem is shown by studying the convergence analysis of the iterative algorithm presented in Section 4. Finally, an example is given to well exemplify our main result in Section 6.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and topological dual E^*, $\langle\cdot,\ \cdot\rangle$ be generalized duality pairing between E^* and E, and $CB(E)$ be the family of all nonempty, bounded and closed subsets of E. Define the *Hausdorff metric* $H(\cdot,\cdot)$ on $CB(E)$ by

$$H(A,B)=\max\{\sup_{u\in A}d(u,B),\sup_{v\in B}d(A,v)\},\quad \forall\ A,\ B\in CB(E),$$

where

$$d(A,v)=\inf_{u\in A}\|u-v\|$$

and

$$d(u,B)=\inf_{v\in B}\|u-v\|.$$

Throughout this paper, unless other stated, $\mathbb{R}$ and $\mathbb{J}$ are denoted by the set of the real numbers and the set $\{1,\ 2,\ 3\}$, respectively. Suppose that for each $i\in\mathbb{J}$, E_i is a real reflexive Banach space with the norm $\|\cdot\|_i$ and the topological dual E_i^*, $\langle\cdot,\cdot\rangle_i$ is the generalized duality between E_i^* and E_i, $H_i(\cdot,\cdot)$ is the Hausdorff metric on $CB(E_i)$ and $CB(E_i^*)$, d_i is the distance between a point and a point set on E_i^* and I_i and I are denoted by the identity mappings defined on E_i and E, respectively. If the norm $\|\cdot\|^*$ on $E_1\times E_2\times E_3$ is defined by

$$\|(u_1,u_2,u_3)\|^*=\|u_1\|_1+\|u_2\|_2+\|u_3\|_3,\quad \forall\ (u_1,u_2,u_3)\in E_1\times E_2\times E_3,$$

then $(E_1\times E_2\times E_3,\ \|\cdot\|^*)$ is a Banach space.

The system of generalized set-valued mixed equilibrium-like problems (in short, **S-GMELP**) is stated as follows:

Suppose that E_1, E_2 and E_3 are real Banach spaces. For each $i\in\mathbb{J}$, let $R_i: E_1\to CB(E_1^*)$, $S_i: E_2\to CB(E_2^*)$, $T_i: E_3\to CB(E_3^*)$ and $F_i: E_i\to CB(E_i)$ be set-valued mappings, let $N_i: E_1^*\times E_2^*\times E_3^*\to E_i^*$, $K_i: E_i\to E_i^*$, $\eta_i: E_i\times E_i\to E_i$ and $M_i: E_i\to E_i$ be single-valued mappings, and let $G_i: E_i^*\times E_i\to\mathbb{R}$ and $B_i: E_i\times E_i\to\mathbb{R}$ be bi-functions. The problem is to find $(u_1,u_2,u_3,x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3,p_1,p_2,p_3)$, where $(u_1,u_2,u_3)\in E_1\times E_2\times E_3$, $(x_i,y_i,z_i)\in R_i(u_1)\times S_i(u_2)\times T_i(u_3)$ and $p_i\in F_i(u_i)$ $(i\in\mathbb{J})$, such that for all $(w_1,w_2,w_3)\in E_1\times E_2\times E_3$ and for each $i\in\mathbb{J}$,

$$\begin{aligned}&G_i(N_i(x_i,y_i,z_i),\eta_i(w_i,u_i))\\&+\langle K_i(u_i),\eta_i(w_i,u_i)\rangle_i\\&+B_i(p_i,M_i(w_i))-B_i(p_i,M_i(u_i))\ge 0,\end{aligned}$$

where for each $i\in\mathbb{J}$, B_i satisfies the following properties:
($\mathbf{A}_1$) for each fixed $p_i\in E_i, u_i\mapsto B_i(p_i,M_i(u_i))$ is convex;
($\mathbf{A}_2$) there exists a constant $b_i>0$ such that

$$\begin{aligned}&B_i(p_i,q_i)+B_i(\hat p_i,\hat q_i)-B_i(p_i,\hat q_i)-B_i(\hat p_i,q_i)\\&\ge -b_i\|p_i-\hat p_i\|_i\cdot\|q_i-\hat q_i\|_i,\\&\quad\forall\ p_i,\ \hat p_i,\ q_i,\ \hat q_i\in E_i;\end{aligned}$$

($\mathbf{A}_3$) $B_i(p_i,0)=B_i(0,q_i)=0,\forall\ p_i,\ q_i\in E_i$.

Remark 1 *It follows from ($\mathbf{A}_2$) and ($\mathbf{A}_3$) that for each $i\in\mathbb{J}$,*
(i) B_i is bounded, that is, there exists a constant $b_i>0$ such that

$$|B_i(p_i,q_i)|\le b_i\|p_i\|_i\cdot\|q_i\|_i,\ \forall\ p_i,\ q_i\in E_i;$$

(ii) $|B_i(p_i,q_i)-B_i(p_i,\hat q_i)|$
$\le b_i\|p_i\|_i\cdot\|q_i-\hat q_i\|_i,\ \forall\ p_i,\ q_i,\ \hat q_i\in E_i$
and
$|B_i(p_i,q_i)-B_i(\hat p_i,q_i)|$
$\le b_i\|q_i\|_i\cdot\|p_i-\hat p_i\|_i,\ \forall\ p_i,\ \hat p_i,\ q_i\in E_i,$
which imply that B_i is continuous in both the first variable and the second variable.

Remark 2 *Let E be a real reflexive Banach space. Many authors (see [8, 12][14]-[17]) usually considered the case that the bi-function $B: E\times E\to\mathbb{R}$ has the following properties:*
($\mathbf{a}_1$) $B(p,q)$ is linear in the first variable;
($\mathbf{a}_2$) $B(p,q)-B(p,\hat q)\le B(p,q-\hat q),\ \forall\ p,\ q,\ \hat q\in E$;
($\mathbf{a}_3$) $B(p,q)$ is bounded;
($\mathbf{a}_4$) for each fixed $p\in E,\ q\mapsto B(p,q)$ is convex.
In addition, Qu [10] considered the case that B satisfies ($\mathbf{A}_2$)-($\mathbf{A}_3$), ($\mathbf{a}_4$) and for each fixed $q\in C$, $p\mapsto B(F(p),q)$ is convex, where C is a nonempty closed convex subset of E and $F: C\to E$ is a single-valued mapping. Zeng-Guu-Yao [13] and Wang-Ding [18] dealt with the case that B satisfies ($\mathbf{a}_1$)-($\mathbf{a}_3$) and ($\mathbf{A}_1$). It's worth mentioning that another distinctive case that B is skew-symmetric and diagonally convex in the second variable was considered in [9]. In [3]-[7][11] authors required that B is a (proper convex) lower semi-continuous single-variable function.

Note that ($\mathbf{a}_1$)-($\mathbf{a}_3$) imply that B is indeed bilinear and ($\mathbf{a}_4$) holds trivially. It is sufficient to show that B is linear in the second variable. First, we claim that

$$\begin{aligned} &B(p,q) + B(p,\hat{q}) \\ &= B(p,q+\hat{q}),\ \forall\, p,\ q,\ \hat{q} \in E \end{aligned} \tag{1}$$

In fact, replacing p by $-p$ in ($\mathbf{a}_2$) and using ($\mathbf{a}_1$), we have

$$\begin{aligned} &B(p,q) - B(p,\hat{q}) \\ &\geq B(p,q-\hat{q}),\ \forall\, p,\ q,\ \hat{q} \in E. \end{aligned} \tag{2}$$

Similarly, (2) implies ($\mathbf{a}_2$) under the condition ($\mathbf{a}_1$). Thus ($\mathbf{a}_2$) is equivalent to (2) under the condition ($\mathbf{a}_1$), and so (1) holds. Second, we claim that for any $k \in \mathbb{R}$,

$$B(p,kq) = kB(p,q),\ \forall\, p,\ q \in E. \tag{3}$$

In fact, combining (1) with $B(p,0) = 0$ for all $p \in E$ that follows from ($\mathbf{a}_3$), we see that (3) holds for any integer k and so does for any rational number k. Now let k be any irrational number and $\{k_n\}$ be a rational number sequence such that $k_n \to k$ as $n \to \infty$. It follows from ($\mathbf{a}_3$) that B is continuous in the second variable. Hence,

$$\begin{aligned} B(p,kq) &= \lim_{n\to\infty} B(p,k_n q) \\ &= \lim_{n\to\infty} k_n B(p,q) \\ &= kB(p,q),\ \forall\, p,\ q \in E. \end{aligned}$$

Finally, we show that B is linear in the second variable by (1) and (3).

If $M = I$ and B satisfies ($\mathbf{a}_1$)-($\mathbf{a}_4$), then B has the properties ($\mathbf{A}_1$)-($\mathbf{A}_3$). Now let $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $B(p,q) = |\sin p| \cdot (\sqrt{1+q^2} - 1)$ and $M = I$. It is easy to see that B has the properties ($\mathbf{A}_1$)-($\mathbf{A}_3$), but B is nonlinear in both the variables.

Some special and related cases are list here:

(a) If $\mathbb{J} = \{1,2\}$, and if for each $i \in \mathbb{J}$, $G_i = \langle\cdot,\cdot\rangle_i$, $R_i = S_i = M_i = F_i = I_i$ and $K_i(u_i) = 0$, then the S-GMELP reduces to the S-MVLIP: to find $(u_1,u_2) \in E_1 \times E_2$ such that for all $(w_1,w_2) \in E_1 \times E_2$ and for each $i \in \mathbb{J}$,

$$\begin{aligned} &\langle N_i(u_1,u_2), \eta_i(w_i,u_i)\rangle_i \\ &+ B_i(u_i,w_i) - B_i(u_i,u_i) \geq 0. \end{aligned}$$

The S-MVLIP was discussed by Kazmi-Khan [17] in Hilbert spaces.

(b) If for each $i \in \mathbb{J}$, $E_i = E$, $E_i^* = E^*$, $G_i = \langle\cdot,\cdot\rangle$, $R_i = R$, $S_i = S$, $F_i = I$, $M_i = M$, $B_i = B$, $N_i(x,y,z) = N(x,y)$, $\eta_i(u,v) = \eta(M(u),M(v))$ and $K_i(u) = \varpi^*$ (where ϖ^* is a given point in E^*), then the S-GMELP becomes the GMVLIP: to find $u \in E$ and $(x,y) \in R(u) \times S(u)$ such that for all $w \in E$,

$$\begin{aligned} &\langle N(x,y) + \varpi^*, \eta(M(w),M(u))\rangle \\ &+ B(u,M(w)) - B(u,M(u)) \geq 0. \end{aligned} \tag{4}$$

The GMVLIP (4) was studied by Ding-Yao-Zeng [16] in Banach spaces, and by Huang-Deng [12], Zeng-Guu-Yao [13], Zeng-Schaible-Yao [14] and Xu-Guo [15] under the case that $\varpi^* = 0$ in Hilbert spaces, respectively.

(c) If for each $i \in \mathbb{J}$, $E_i = E$, $E_i^* = E^*$ and C is a nonempty (closed) convex subset of E, if $G_i = \langle\cdot,\cdot\rangle$, $R_i = R$, $S_i = S$, $F_i = F$, $M_i = I$, $\eta_i = \eta$, $B_i = B$ and $N_i(x,y,z) = N(x,y)$, and if R, S and F are single-valued mappings defined on C, both η and B are defined on $E \times E$ or $C \times C$ and $K_i(u) = \varpi^*$ (where ϖ^* is a given point in E^*), then the S-GMELP reduces to the MVLIP: to find $u \in C$ such that for all $w \in C$,

$$\begin{aligned} &\langle N(R(u),S(u)) + \varpi^*, \eta(w,u)\rangle \\ &+ B(F(u),w) - B(F(u),u) \geq 0. \end{aligned} \tag{5}$$

The MVLIP (5) was discussed by Ding [8, 9] and Qu [10] in Banach space.

(d) If for each $i \in \mathbb{J}$, $E_i = E$, $E_i^* = E^*$, $G_i = \langle\cdot,\cdot\rangle$, $R_i = R$, $S_i = S$, $M_i = I$, $\eta_i = \eta$, $K_i(u) = 0$, $N_i(x,y,z) = N(x,y)$ and $B_i(u,w) = B(w)$, then the S-GMELP reduces to the GMVLIP: to find $u \in E$ and $(x,y) \in R(u) \times S(u)$ such that for all $w \in E$,

$$\langle N(x,y), \eta(w,u)\rangle + B(w) - B(u) \geq 0. \tag{6}$$

The GMVLIP (6) was studied in [11] by discussing the convergence analysis of a predictor-corrector iterative algorithm in Hilbert spaces.

(e) If for each $i \in \mathbb{J}$, $E_i = E$, $E_i^* = E^*$ and C is a nonempty convex subset of E, and if $G_i = \langle\cdot,\cdot\rangle$, $R_i = R$, $S_i = S$, $M_i = I$, $\eta_i = \eta$, $B_i(u,w) = B(w)$, $N_i(x,y,z) = N(x,y)$ and $K_i(u) = 0$, and if R, S and B are single-valued mappings defined on C and η is defined on $C \times C$, then the S-GMELP reduces to the MVLIP: to find $u \in C$ such that for all $w \in C$,

$$\begin{aligned} &\langle N(R(u),S(u)), \eta(w,u)\rangle \\ &+ B(w) - B(u) \geq 0. \end{aligned} \tag{7}$$

In [7] the existence and uniqueness of solutions of the MVLIP (7) were directly proved and a general algorithm to approximate the solution of the MVLIP (7) was proposed by using an auxiliary minimization problem in Banach spaces.

(f) If for each $i \in \mathbb{J}$, $E_i = E$, $E_i^* = E^*$ and C is a nonempty convex subset of E, and if

$G_i = \langle\cdot,\cdot\rangle,\ R_i = R,\ S_i = S,\ M_i = I,\ \eta_i = \eta,\ N_i(x,y,z) = x - y$ and $B_i(u,w) = B(w)$, and if $R,\ S$ and B are single-valued mappings defined on C, η is defined on $C\times C$ and $K_i(u) = \varpi^*$ (where ϖ^* is a given point in E^*), then the S-GMELP reduces to the MVLIP: to find $u \in C$ such that for all $w \in C$,

$$\begin{aligned}&\langle R(u) - S(u) + \varpi^*, \eta(w,u)\rangle\\ &+ B(w) - B(u) \geq 0.\end{aligned} \tag{8}$$

The MVLIP (8) was discussed by Zeng [5] and Ansari-Yao [6] in Hilbert spaces, and by Ding [3] and Chen-Liu [4] in Banach spaces, respectively.

Some other related cases were discussed by Wang-Ding [18] and Fang-Huang [19] in Banach spaces and by Chadli-Yao [20] in Hausdorff topological linear spaces, respectively.

Definition 3 *Let E be a real Banach space, and let $S : E \to CB(E^*)$, $\eta : E \times E \to E$, $M : E \to E$ and $g : E \to E^*$ be mappings.*

(i) S is said to be s-H- Lipschitz continuous , if there exists a constant $s > 0$ such that

$$H(S(u), S(v)) \leq s\|u - v\|,\ \forall\, u,\ v \in E.$$

(ii) η is said to be ξ- Lipschitz continuous, if there exists a constant $\xi > 0$ such that

$$\|\eta(u,v)\| \leq \xi\|u - v\|,\ \forall\, u,\ v \in E.$$

(iii) M is said to be m-Lipschitz continuous , if there exists a constant $m > 0$ such that

$$\|M(u) - M(v)\| \leq m\|u - v\|,\ \forall\, u,\ v \in E.$$

(iv)([16]) g is said to be τ-η-strongly monotone, if there exists a constant $\tau > 0$ such that

$$\langle g(u) - g(v), \eta(u,v)\rangle \geq \tau\|u - v\|^2,\ \forall\, u,\ v \in E.$$

Remark 4 *If $\eta(u,v) = u-v$ (resp., $\eta(u,v) = v-u$), the η-strong monotonicity of g reduces to the strong monotonicity of g (resp., $-g$).*

Definition 5 *([18]) Let $E_1,\ E_2$ and E_3 be Banach spaces, and $N :\ E_1^* \times E_2^* \times E_3^* \to E_1^*$ be a mapping. N is said to be (μ,θ,ϑ)-mixed Lipschitz continuous, if there exist three positive constants $\mu,\ \theta,\ \vartheta$ such that*

$$\begin{aligned}&\|N(x,y,z) - N(\hat{x},\hat{y},\hat{z})\|_1\\ &\leq \mu\|x - \hat{x}\|_1 + \theta\|y - \hat{y}\|_2 + \vartheta\|z - \hat{z}\|_3,\\ &\ \forall\,(x,y,z),\ (\hat{x},\hat{y},\hat{z}) \in E_1^* \times E_2^* \times E_3^*.\end{aligned}$$

Lemma 6 *([21]) Let X be a nonempty closed convex subset of a Hausdorff linear topological space E and $\phi,\ \psi : X \times X \to \mathbb{R}$ be bi-functions satisfying the following conditions :*
(i) $\psi(x,y) \leq \phi(x,y),\ \forall\, x,\ y \in X$ and $\psi(x,x) \geq 0,\ \forall\, x \in X$;
(ii) for each $x \in X$, $\phi(x,y)$ is upper semi-continuous with respect to y;
(iii) for each $y \in X$, the set $\{x \in X : \psi(x,y) < 0\}$ is convex ;
(iv) there exists a nonempty compact set $\Omega \subset X$ and $x_0 \in \Omega$ such that $\psi(x_0,y) < 0,\ \forall\, y \in X \setminus \Omega$. Then there exists an $\bar{y} \in \Omega$ such that $\phi(x,\bar{y}) \geq 0,\ \forall\, x \in X$.

We shall also make use of the following result which is a variation of Lemma 1 in [22] and is also noted implicitly in [23].

Lemma 7 *Let $(E,\ d)$ be a complete metric space and $S : E \to CB(E)$ be a set-valued mapping. Then for any $\varepsilon > 0$, any $u,\ v \in E$, and any $x \in S(u)$, there exists $y \in S(v)$ such that*

$$d(x,y) \leq (1+\varepsilon)H(S(u), S(v)).$$

Remark 8 *Let E be a normal linear space. If $f : E \to \mathbb{R}$ is concave and upper semi-continuous, then f is weakly upper semi-continuous.*

3 A system of auxiliary problems

Now a system of auxiliary problems (in short, **S-AP**) for the S-GMELP is given below:

Suppose that $E_1,\ E_2$ and E_3 are real reflexive Banach spaces. For each $i \in \mathbb{J}$, let $R_i : E_1 \to CB(E_1^*),\ S_i : E_2 \to CB(E_2^*),\ T_i : E_3 \to CB(E_3^*)$ and $F_i : E_i \to CB(E_i)$ be set-valued mappings, let $K_i,\ g_i : E_i \to E_i^*,\ M_i : E_i \to E_i,\ N_i : E_1^* \times E_2^* \times E_3^* \to E_i^*$ and $\eta_i : E_i \times E_i \to E_i$ be single-valued mappings, and let $G_i : E_i^* \times E_i \to \mathbb{R}$ and $B_i : E_i \times E_i \to \mathbb{R}$ be bi-functions. Given $(u_1,u_2,u_3) \in E_1 \times E_2 \times E_3$, for each $i \in \mathbb{J},\ (x_i,y_i,z_i) \in R_i(u_1) \times S_i(u_2) \times T_i(u_3)$ and $p_i \in F_i(u_i)$, the S-AP is to find $(v_1,v_2,v_3) \in E_1 \times E_2 \times E_3$ such that for all $(w_1,w_2,w_3) \in E_1 \times E_2 \times E_3$ and for each $i \in \mathbb{J}$,

$$\begin{aligned}&\langle g_i(v_i) - g_i(u_i), \eta_i(w_i,v_i)\rangle_i\\ &+\rho G_i(N_i(x_i,y_i,z_i), \eta_i(w_i,v_i))\\ &+\rho\langle K_i(u_i), \eta_i(w_i,v_i)\rangle_i\\ &+\rho B_i(p_i, M_i(w_i)) - \rho B_i(p_i, M_i(v_i))\\ &\geq 0,\end{aligned}$$

where $\rho > 0$ is a constant.

Theorem 9 *Suppose that for each $i \in \mathbb{J}$, (**A**$_1$)-(**A**$_3$) and the following conditions hold:*
*(**B**$_1$) K_i and $g_i - K_i$ are λ_i- and γ_i- Lipschitz continuous, respectively, and g_i is τ_i-η_i- strongly monotone;*
*(**B**$_2$) M_i* is *m_i-Lipschitz continuous;*
*(**B**$_3$) η_i* is *ξ_i-Lipschitz continuous;*
*(**B**$_4$) $\eta_i(w_i, v_i) = -\eta_i(v_i, w_i)$ and $\eta_i(e_i, v_i) - \eta_i(e_i, w_i) = \eta_i(v_i, w_i)$, $\forall\, e_i,\ v_i,\ w_i \in E_i$;*
*(**B**$_5$) for any fixed $w_i \in E_i$ and $w_i^* \in E_i^*$, both $v_i \mapsto G_i(w_i^*, \eta_i(w_i, v_i))$ and $v_i \mapsto \langle w_i^*, \eta_i(w_i, v_i)\rangle_i$ are concave;*
*(**B**$_6$) $G_i(\cdot, -v_i) = -G_i(\cdot, v_i)$, $\forall\, v_i \in E_i$, and there exists a constant $\kappa_i > 0$ such that*

$$\begin{aligned} &|G_i(v_i^*, v_i) - G_i(\hat{v}_i^*, v_i)| \\ &\le \kappa_i \|v_i\|_i \cdot \|v_i^* - \hat{v}_i^*\|_i, \forall\, v_i \in E_i,\ v_i^*,\ \hat{v}_i^* \in E_i^*, \\ &|G_i(v_i^*, v_i) - G_i(v_i^*, \hat{v}_i)| \\ &\le \kappa_i \|v_i^*\|_i \cdot \|v_i - \hat{v}_i\|_i, \forall\, v_i,\ \hat{v}_i \in E_i,\ v_i^* \in E_i^*. \end{aligned}$$

Then the S-AP has a unique solution.

Remark 10 *The conditions (**B**$_4$)-(**B**$_6$) imply that*
(i) $\eta_i(v_i, v_i) = 0$, $\forall v_i \in E_i$;
(ii) $G_i(v_i^*, 0) = 0$, $\forall\, v_i^* \in E_i^*$;
(iii) $|G_i(v_i^*, v_i)| \le \kappa_i \|v_i\|_i \cdot \|v_i^*\|_i$, $\forall\, v_i \in E_i$, $v_i^* \in E_i^*$;
(iv) for any fixed $w_i \in E_i$ and $w_i^* \in E_i^*$, both $v_i \mapsto G_i(w_i^*, \eta_i(v_i, w_i))$ and $v_i \mapsto \langle w_i^*,\ \eta_i(v_i, w_i)\rangle_i$ are convex.

Remark 11 *It follows from (**B**$_1$) and (**B**$_3$) that g_i is β_i-Lipschitz continuous and*

$$\tau_i \le \beta_i \xi_i \le \xi_i(\lambda_i + \gamma_i).$$

In fact, for any $u_i,\ v_i \in E_i$,

$$\begin{aligned} &\|g_i(u_i) - g_i(v_i)\|_i \\ &\le \|(g_i - K_i)(u_i) - (g_i - K_i)(v_i)\|_i \\ &\quad + \|K_i(u_i) - K_i(v_i)\|_i \\ &\le (\lambda_i + \gamma_i)\|u_i - v_i\|_i. \end{aligned}$$

Letting

$$\beta_i = \inf\{\beta_i^* :\ \|g_i(u_i) - g_i(v_i)\|_i \le \beta_i^* \|u_i - v_i\|_i\},$$

*we have that $\beta_i \le \lambda_i + \gamma_i$. In view of (**B**$_1$) and (**B**$_3$), for any $u_i,\ v_i \in E_i$, the following inequalities hold:*

$$\begin{aligned} &\tau_i \|u_i - v_i\|_i^2 \\ &\le \langle g_i(u_i) - g_i(v_i), \eta_i(u_i, v_i)\rangle_i \\ &\le \|g_i(u_i) - g_i(v_i)\|_i \cdot \|\eta_i(u_i, v_i)\|_i \\ &\le \beta_i \xi_i \|u_i - v_i\|_i^2, \end{aligned}$$

which implies that $\tau_i \le \beta_i \xi_i$ and $\beta_i > 0$. Thus g_i is β_i-Lipschitz continuous and $\tau_i \le \beta_i \xi_i \le \xi_i(\lambda_i + \gamma_i)$.

Proof of theorem 9. For each $i \in \mathbb{J}$, define the mappings $\phi_i,\ \psi_i : E_i \times E_i \to \mathbb{R}$ by

$$\begin{aligned} &\phi_i(w_i, v_i) \\ &= \langle g_i(w_i) - g_i(u_i), \eta_i(w_i, v_i)\rangle_i \\ &\quad + \rho G_i(N_i(x_i, y_i, z_i), \eta_i(w_i, v_i)) \\ &\quad + \rho\langle K_i(u_i), \eta_i(w_i, v_i)\rangle_i \\ &\quad + \rho B_i(p_i, M_i(w_i)) - \rho B_i(p_i, M_i(v_i)), \\ &\psi_i(w_i, v_i) \\ &= \langle g_i(v_i) - g_i(u_i), \eta_i(w_i, v_i)\rangle_i \\ &\quad + \rho G_i(N_i(x_i, y_i, z_i), \eta_i(w_i, v_i))) \\ &\quad + \rho\langle K_i(u_i), \eta_i(w_i, v_i)\rangle_i \\ &\quad + \rho B_i(p_i, M_i(w_i)) - \rho B_i(p_i, M_i(v_i)), \end{aligned}$$

respectively. We complete this proof by three steps.

Step 1. Show that for each given $(u_1, u_2, u_3) \in E_1 \times E_2 \times E_3$, $(x_i, y_i, z_i) \in R_i(u_1) \times S_i(u_2) \times T_i(u_3)$, and $p_i \in F_i(u_i)$ $(i \in \mathbb{J})$, ϕ_i and ψ_i satisfy all the conditions of Lemma 6 in the weak topology.

(i) Show that $\psi_i(w_i, v_i) \le \phi_i(w_i, v_i)$ and $\psi_i(w_i, w_i) \ge 0$, $\forall\, w_i,\ v_i \in E_i$. Indeed, since g_i is τ_i-η_i-strongly monotone, it is clear that the two inequalities above hold.

(ii) Show that for each $w_i \in E_i$, $v_i \mapsto \phi_i(w_i, v_i)$ is weakly upper semi-continuous. In fact, by (**B**$_3$)-(**B**$_4$) and (**B**$_6$), for each $w_i,\ v_i,\ \hat{v}_i \in E_i$, we have

$$\begin{aligned} &|\langle g_i(w_i) - g_i(u_i), \eta_i(w_i, v_i)\rangle_i \\ &\quad - \langle g_i(w_i) - g_i(u_i), \eta_i(w_i, \hat{v}_i)\rangle_i| \\ &= |\langle g_i(w_i) - g_i(u_i), \eta_i(v_i, \hat{v}_i)\rangle_i| \\ &\le \xi_i \|g_i(w_i) - g_i(u_i)\|_i \cdot \|v_i - \hat{v}_i\|_i, \end{aligned}$$

$$\begin{aligned} &|G_i(N_i(x_i, y_i, z_i), \eta_i(w_i, v_i)) \\ &\quad - G_i(N_i(x_i, y_i, z_i), \eta_i(w_i, \hat{v}_i))| \\ &\le \kappa_i \|N_i(x_i, y_i, z_i)\|_i \cdot \|\eta_i(w_i, v_i) - \eta_i(w_i, \hat{v}_i)\|_i \\ &= \kappa_i \|N_i(x_i, y_i, z_i)\|_i \cdot \|\eta_i(v_i, \hat{v}_i)\|_i \\ &\le \kappa_i \xi_i \|N_i(x_i, y_i, z_i)\|_i \cdot \|v_i - \hat{v}_i\|_i, \end{aligned}$$

and

$$\begin{aligned} &|\langle K_i(u_i), \eta_i(w_i, v_i)\rangle_i - \langle K_i(u_i), \eta_i(w_i, \hat{v}_i)\rangle_i| \\ &\le \xi_i \|K_i(u_i)\|_i \cdot \|v_i - \hat{v}_i\|_i. \end{aligned}$$

By Remark 1 (ii) and (**B**$_3$), for each $v_i,\ \hat{v}_i \in E_i$, we have

$$\begin{aligned} &|B_i(p_i, M_i(v_i)) - B_i(p_i, M_i(\hat{v}_i))| \\ &\le b_i m_i \|p_i\|_i \cdot \|v_i - \hat{v}_i\|_i. \end{aligned}$$

Thus $v_i \mapsto \phi_i(w_i, v_i)$ is continuous. Noting that $v_i \mapsto \phi_i(w_i, v_i)$ is concave by (**A**$_1$) and (**B**$_5$), we see that $v_i \mapsto \phi_i(w_i, v_i)$ is weakly upper semi-continuous by Remark 8.

(iii) Show that for each fixed $v_i \in E_i$, the set $C_i = \{w_i \in E_i : \psi_i(w_i, v_i) < 0\}$ is convex. If $C_i = \emptyset$, then the i-th inequality of the S-AP holds trivially. Hence we only discuss the case that $C_i \neq \emptyset$. It follows from ($\mathbf{A}_1$) and Remark 10 (iv) that $w_i \mapsto \psi_i(w_i, v_i)$ is convex. Thus C_i is convex.

(iv) Show that there exists a nonempty weakly compact set $\Omega_i \subset E_i$ and $\bar{w}_i \in \Omega$ such that $\psi_i(\bar{w}_i, v_i) < 0,\ \forall\, v_i \in E_i \setminus \Omega_i$. Now take

$$\begin{aligned}\delta_i = \tau_i^{-1}[&\xi_i\|g_i(u_i) + \rho K_i(u_i)\|_i\\ &+ \rho\xi_i\kappa_i\|N_i(x_i, y_i, z_i)\|_i + \rho b_i m_i\|p_i\|_i],\\ \Omega_i = \{&w_i \in E_i : \|w_i\|_i \leq \delta_i\}.\end{aligned}$$

Then Ω_i is a weakly compact subset of E_i. For any fixed $v_i \in E_i \setminus \Omega_i$, take $\bar{w}_i = 0 \in \Omega_i$. It follows from ($\mathbf{B}_2$)-($\mathbf{B}_3$), Remark 1(ii) and Remark 10 (ii) and (iv) that

$$\begin{aligned}&\psi_i(\bar{w}_i, v_i) = \psi_i(0, v_i)\\ &= -\langle g_i(0) - g_i(v_i), \eta_i(0, v_i)\rangle_i\\ &\quad + \langle g_i(u_i) + \rho K_i(u_i), \eta_i(0, v_i)\rangle_i\\ &\quad + \rho G_i(N_i(x_i, y_i, z_i), \eta_i(0, v_i))\\ &\quad + \rho[B_i(p_i, M_i(0)) - B_i(p_i, M_i(v_i))]\\ &\leq -\tau_i\|v_i\|_i^2 + (\xi_i\|g_i(u_i) + \rho K_i(u_i)\|_i\\ &\quad + \rho\xi_i\kappa_i\|N_i(x_i, y_i, z_i)\|_i + \rho b_i m_i\|p_i\|_i) \cdot \|v_i\|_i\\ &= -\tau_i\|v_i\|_i(\|v_i\|_i - \delta_i)\\ &< 0.\end{aligned}$$

Therefore, by Lemma 6, there exists a $\bar{v}_i \in E_i$ such that for all $w_i \in E_i$, $\phi_i(w_i, \bar{v}_i) \geq 0$, that is,

$$\begin{aligned}&\langle g_i(w_i) - g_i(u_i), \eta_i(w_i, \bar{v}_i)\rangle_i\\ &+ \rho G_i(N_i(x_i, y_i, z_i), \eta_i(w_i, \bar{v}_i))\\ &+ \rho\langle K_i(u_i), \eta_i(w_i, \bar{v}_i)\rangle_i\\ &+ \rho[B_i(p_i, M_i(w_i)) - B_i(p_i, M_i(\bar{v}_i))]\\ &\geq 0.\end{aligned} \tag{9}$$

Step 2 Show that when $C_i \neq \emptyset$, there exists a solution of the i-th inequality of the S-AP.

For each $i \in \mathbb{J}$, arbitrary $t \in (0, 1]$ and any fixed $w_i \in E_i$, let $w_i(t) = tw_i + (1-t)\bar{v}_i$. Replacing w_i by $w_i(t)$ in (9) and applying Remark 10 (i), (iv) and ($\mathbf{A}_1$), we have

$$\begin{aligned}0 &\leq \langle g_i(w_i(t)) - g_i(u_i), \eta_i(tw_i + (1-t)\bar{v}_i, \bar{v}_i)\rangle_i\\ &\quad + \rho G_i(N_i(x_i, y_i, z_i), \eta_i(tw_i + (1-t)\bar{v}_i, \bar{v}_i))\\ &\quad + \rho\langle K_i(u_i), \eta_i(tw_i + (1-t)\bar{v}_i, \bar{v}_i)\rangle_i\\ &\quad + \rho B_i(p_i, M_i(tw_i + (1-t)\bar{v}_i))\\ &\quad - \rho B_i(p_i, M_i(\bar{v}_i))\\ &\leq t[\langle g_i(w_i(t)) - g_i(u_i), \eta_i(w_i, \bar{v}_i)\rangle_i\\ &\quad + \rho G_i(N_i(x_i, y_i, z_i), \eta_i(w_i, \bar{v}_i))\\ &\quad + \rho\langle K_i(u_i), \eta_i(w_i, \bar{v}_i)\rangle_i\\ &\quad + \rho B_i(p_i, M_i(\bar{v}_i)) - \rho B_i(p_i, M_i(w_i))].\end{aligned}$$

Hence,

$$\begin{aligned}&\langle g_i(w_i(t)) - g_i(u_i), \eta_i(w_i, \bar{v}_i)\rangle_i\\ &+ \rho G_i(N_i(x_i, y_i, z_i), \eta_i(w_i, \bar{v}_i))\\ &+ \rho\langle K_i(u_i), \eta_i(w_i, \bar{v}_i)\rangle_i\\ &+ \rho B_i(p_i, M_i(w_i)) - \rho B_i(p_i, M_i(\bar{v}_i))\\ &\geq 0.\end{aligned} \tag{10}$$

Letting $t \to 0^+$ in (10) and applying Remark 11, we obtain

$$\begin{aligned}&\langle g_i(\bar{v}_i) - g_i(u_i), \eta_i(w_i, \bar{v}_i)\rangle_i\\ &+\rho G_i(N_i(x_i, y_i, z_i), \eta_i(w_i, \bar{v}_i))\\ &+\rho\langle K_i(u_i), \eta_i(w_i, \bar{v}_i)\rangle_i\\ &+\rho B_i(p_i, M_i(w_i)) - \rho B_i(p_i, M_i(\bar{v}_i))\\ &\geq 0.\end{aligned}$$

Therefore, the existence of solutions of the S-AP is shown.

Step 3 Show that the solution of the S-AP is unique.

Let $(\bar{v}_1, \bar{v}_2, \bar{v}_3)$, $(\hat{v}_1, \hat{v}_2, \hat{v}_3) \in E_1 \times E_2 \times E_3$ be two solutions of the S-AP. Then for each $i \in \mathbb{J}$ and for all $(w_1, w_2, w_3) \in E_1 \times E_2 \times E_3$,

$$\begin{aligned}&\langle g_i(\bar{v}_i), \eta_i(w_i, \bar{v}_i)\rangle_i\\ &\geq \langle g_i(u_i), \eta_i(w_i, \bar{v}_i)\rangle_i\\ &- \rho G_i(N_i(x_i, y_i, z_i), \eta_i(w_i, \bar{v}_i))\\ &- \rho\langle K_i(u_i), \eta_i(w_i, \bar{v}_i)\rangle_i\\ &+ \rho B_i(p_i, M_i(\bar{v}_i)) - \rho B_i(p_i, M_i(w_i)),\end{aligned} \tag{11}$$

and

$$\begin{aligned}&\langle g_i(\hat{v}_i), \eta_i(w_i, \hat{v}_i)\rangle_i\\ &\geq \langle g_i(u_i), \eta_i(w_i, \hat{v}_i)\rangle_i\\ &- \rho G_i(N_i(x_i, y_i, z_i), \eta_i(w_i, \hat{v}_i))\\ &- \rho\langle K_i(u_i), \eta_i(w_i, \hat{v}_i)\rangle_i\\ &+ \rho B_i(p_i, M_i(\hat{v}_i)) - \rho B_i(p_i, M_i(w_i)).\end{aligned} \tag{12}$$

By taking $w_i = \hat{v}_i$ in (11) and $w_i = \bar{v}_i$ in (12) and utilizing ($\mathbf{B}_4$) and ($\mathbf{B}_6$),

$$\begin{aligned}&\langle g_i(\bar{v}_i) - g_i(\hat{v}_i), \eta_i(\hat{v}_i, \bar{v}_i)\rangle_i\\ &\geq -\rho[G_i(N_i(x_i, y_i, z_i), \eta_i(\bar{v}_i, \hat{v}_i))\\ &\quad + G_i(N_i(x_i, y_i, z_i), \eta_i(\hat{v}_i, \bar{v}_i))]\\ &\quad - \rho\langle K_i(u_i), \eta_i(\bar{v}_i, \hat{v}_i) + \eta_i(\hat{v}_i, \bar{v}_i)\rangle_i\\ &= \rho[G_i(N_i(x_i, y_i, z_i), \eta_i(\hat{v}_i, \bar{v}_i))\\ &\quad - G_i(N_i(x_i, y_i, z_i), \eta_i(\hat{v}_i, \bar{v}_i))]\\ &= 0.\end{aligned}$$

Since g_i is τ_i-η_i-strongly monotone, the following inequality holds:

$$\begin{aligned}&\tau_i\|\bar{v}_i - \hat{v}_i\|_i^2\\ &\leq \langle g_i(\bar{v}_i) - g_i(\hat{v}_i), \eta_i(\bar{v}_i, \hat{v}_i)\rangle_i\\ &\leq 0,\end{aligned}$$

which implies that $\bar{v}_i = \hat{v}_i$. This completes the proof of Theorem 9. □

4 An iterative algorithm for S-GMELP

On the basis of Theorem 9, now an iterative algorithm for solving the S-GMELP is constructed in real reflexive Banach spaces. For each $i \in \mathbb{J}$ and given $(u_1^0, u_2^0, u_3^0) \in E_1 \times E_2 \times E_3$, $(x_i^0, y_i^0, z_i^0) \in R_i(u_1^0) \times S_i(u_2^0) \times T_i(u_3^0)$ and $p_i^0 \in F_i(u_i^0)$, the unique solution $(u_1^1, u_2^1, u_3^1) \in E_1 \times E_2 \times E_3$ of the S-AP satisfies that for all $(w_1, w_2, w_3) \in E_1 \times E_2 \times E_3$ and for each $i \in \mathbb{J}$,

$$\begin{aligned}&\langle g_i(u_i^1) - g_i(u_i^0), \eta_i(w_i, u_i^1)\rangle_i\\&+\rho G_i(N_i(x_i^0, y_i^0, z_i^0), \eta_i(w_i, u_i^1))\\&+\rho\langle K_i(u_i^0), \eta_i(w_i, u_i^1)\rangle_i\\&+\rho B_i(p_i^0, M_i(w_i)) - \rho B_i(p_i^0, M_i(u_i^1))\\&\geq 0.\end{aligned}$$

By Lemma 7, there exist $(x_i^1, y_i^1, z_i^1) \in R_i(u_1^1) \times S_i(u_2^1) \times T_i(u_3^1)$ and $p_i^1 \in F_i(u_i^1)$ such that

$$\begin{aligned}&\|x_i^0 - x_i^1\|_1 \leq (1+1)H_1(R_i(u_1^0), R_i(u_1^1)),\\&\|y_i^0 - y_i^1\|_2 \leq (1+1)H_2(S_i(u_2^0), S_i(u_2^1)),\\&\|z_i^0 - z_i^1\|_3 \leq (1+1)H_3(T_i(u_3^0), T_i(u_3^1)),\\&\|p_i^0 - p_i^1\|_i \leq (1+1)H_i(F_i(u_i^0), F_i(u_i^1)).\end{aligned}$$

By induction, an iterative algorithm for solving the S-GMELP is established in real reflexive Banach spaces E_1, E_2 and E_3 as follows:

Algorithm I *For each $i \in \mathbb{J}$ and for any given $(u_1^0, u_2^0, u_3^0) \in E_1 \times E_2 \times E_3$, $(x_i^0, y_i^0, z_i^0) \in R_i(u_1^0) \times S_i(u_2^0) \times T_i(u_3^0)$ and $p_i^0 \in F_i(u_i^0)$, there exists a sequence $\{Q_n\}_{n=1}^\infty$, where*

$$\begin{aligned}Q_n = (&u_1^n, u_2^n, u_3^n, x_1^n, x_2^n, x_3^n, y_1^n, y_2^n, y_3^n,\\&z_1^n, z_2^n, z_3^n, p_1^n, p_2^n, p_3^n),\end{aligned}$$

such that $x_i^n \in R_i(u_1^n)$, $y_i^n \in S_i(u_2^n)$, $z_i^n \in T_i(u_3^n)$, $p_i^n \in F_i(u_i^n)$ and

$$\begin{aligned}&\|x_i^n - x_i^{n+1}\|_1 \leq (1 + \tfrac{1}{n+1})H_1(R_i(u_1^n), R_i(u_1^{n+1})),\\&\|y_i^n - y_i^{n+1}\|_2 \leq (1 + \tfrac{1}{n+1})H_2(S_i(u_2^n), S_i(u_2^{n+1})),\\&\|z_i^n - z_i^{n+1}\|_3 \leq (1 + \tfrac{1}{n+1})H_3(T_i(u_3^n), T_i(u_3^{n+1})),\\&\|p_i^n - p_i^{n+1}\|_i \leq (1 + \tfrac{1}{n+1})H_i(F_i(u_i^n), F_i(u_i^{n+1})),\end{aligned} \tag{13}$$

and for all $(w_1, w_2, w_3) \in E_1 \times E_2 \times E_3$ and for each $i \in \mathbb{J}$,

$$\begin{aligned}&\langle g_i(u_i^{n+1}) - g_i(u_i^n), \eta_i(w_i, u_i^{n+1})\rangle_i\\&+ \rho G_i(N_i(x_i^n, y_i^n, z_i^n), \eta_i(w_i, u_i^{n+1}))\\&+ \rho\langle K_i(u_i^n), \eta_i(w_i, u_i^{n+1})\rangle_i\\&+ \rho B_i(p_i^n, M_i(w_i)) - \rho B_i(p_i^n, M_i(u_i^{n+1}))\\&\geq 0,\end{aligned} \tag{14}$$

where $\rho > 0$ is a constant.

5 Existence of solutions of S-GMELP and convergence analysis

In this section, we shall show that the sequence $\{Q_n\}_{n=1}^\infty$ generalized by **Algorithm I** strongly converges to some solution of the S-GMELP.

Theorem 12 *Let E_1, E_2 and E_3 be real reflexive Banach spaces. Suppose that for each $i \in \mathbb{J}$, **(A_1)-(A_3)**, **(B_1)-(B_6)** and the following conditions hold:*
(C_1) *N_i is $(\mu_i, \theta_i, \vartheta_i)$-mixed Lipschitz continuous;*
(C_2) *R_i, S_i, T_i and F_i are r_i-H_1-, s_i-H_2-, t_i-H_3- and f_i-H_i-Lipschitz continuous, respectively.*
If there exists a constant $\rho > 0$ such that

$$\sigma = \max\{\sigma_1,\ \sigma_2,\ \sigma_3\} < 1, \tag{15}$$

where for each $i \in \mathbb{J}$,

$$\begin{cases}\sigma_i = \tau_i^{-1}\xi_i(\gamma_i + |1-\rho|\lambda_i) + \rho\tau_i^{-1}\alpha_i,\\ \alpha_1 = b_1 f_1 m_1 + \sum\limits_{i=1}^{3}\kappa_i\xi_i\mu_i r_i,\\ \alpha_2 = b_2 f_2 m_2 + \sum\limits_{i=1}^{3}\kappa_i\xi_i\theta_i s_i,\\ \alpha_3 = b_3 f_3 m_3 + \sum\limits_{i=1}^{3}\kappa_i\xi_i\vartheta_i t_i,\end{cases} \tag{16}$$

*then for each $i \in \mathbb{J}$, the sequence $\{Q_n\}_{n=1}^\infty$ generalized by **Algorithm I** strongly converges to $Q = (\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{p}_1, \bar{p}_2, \bar{p}_3)$, and Q is a solution of the S-GMELP, where $(\bar{u}_1, \bar{u}_2, \bar{u}_3) \in E_1 \times E_2 \times E_3$ and for each $i \in \mathbb{J}$, $(\bar{x}_i, \bar{y}_i, \bar{z}_i) \in R_i(\bar{u}_1) \times S_i(\bar{u}_2) \times T_i(\bar{u}_3)$ and $\bar{p}_i \in F_i(\bar{u}_i)$.*

Proof. It follows from (14) that for any $(w_1, w_2, w_3) \in E_1 \times E_2 \times E_3$ and for each $i \in \mathbb{J}$,

$$\begin{aligned}&\langle g_i(u_i^n) - g_i(u_i^{n-1}), \eta_i(w_i, u_i^n)\rangle_i\\&\quad+ \rho G_i(N_i(x_i^{n-1}, y_i^{n-1}, z_i^{n-1}), \eta_i(w_i, u_i^n))\\&\quad+ \rho\langle K_i(u_i^{n-1}), \eta_i(w_i, u_i^n)\rangle_i\\&\quad+ \rho B_i(p_i^{n-1}, M_i(w_i)) - \rho B_i(p_i^{n-1}, M_i(u_i^n))\\&\quad\geq 0,\end{aligned} \tag{17}$$

and

$$\begin{aligned}&\langle g_i(u_i^{n+1}) - g_i(u_i^n), \eta_i(w_i, u_i^{n+1})\rangle_i\\&+ \rho G_i(N_i(x_i^n, y_i^n, z_i^n), \eta_i(w_i, u_i^{n+1}))\\&+ \rho\langle K_i(u_i^n), \eta_i(w_i, u_i^{n+1})\rangle_i\\&+ \rho B_i(p_i^n, M_i(w_i)) - \rho B_i(p_i^n, M_i(u_i^{n+1}))\\&\geq 0.\end{aligned} \tag{18}$$

Taking $w_i = u_i^{n+1}$ in (17) and $w_i = u_i^n$ in (18), respectively, we get that for each $i \in \mathbb{J}$,

$$\begin{aligned}&\langle g_i(u_i^n) - g_i(u_i^{n-1}), \eta_i(u_i^{n+1}, u_i^n)\rangle_i\\&+ \rho G_i(N_i(x_i^{n-1}, y_i^{n-1}, z_i^{n-1}), \eta_i(u_i^{n+1}, u_i^n))\\&+ \rho\langle K_i(u_i^{n-1}), \eta_i(u_i^{n+1}, u_i^n)\rangle_i\\&+ \rho[B_i(p_i^{n-1}, M_i(u_i^{n+1})) - B_i(p_i^{n-1}, M_i(u_i^n))]\\&\geq 0,\end{aligned} \tag{19}$$

and

$$\begin{aligned}&\langle g_i(u_i^{n+1}) - g_i(u_i^n), \eta_i(u_i^n, u_i^{n+1})\rangle_i\\&+ \rho G_i(N_i(x_i^n, y_i^n, z_i^n), \eta_i(u_i^n, u_i^{n+1}))\\&+ \rho\langle K_i(u_i^n), \eta_i(u_i^n, u_i^{n+1})\rangle_i\\&+ \rho[B_i(p_i^n, M_i(u_i^n)) - B_i(p_i^n, M_i(u_i^{n+1}))]\\&\geq 0.\end{aligned} \tag{20}$$

Combining (19)-(20) with ($\mathbf{B}_3$), we have

$$\begin{aligned}&\langle g_i(u_i^n) - g_i(u_i^{n+1}), \eta_i(u_i^n, u_i^{n+1})\rangle_i\\\leq\ &\langle g_i(u_i^{n-1}) - g_i(u_i^n), \eta_i(u_i^n, u_i^{n+1})\rangle_i\\&+ \rho\langle K_i(u_i^n) - K_i(u_i^{n-1}), \eta_i(u_i^n, u_i^{n+1})\rangle_i\\&+ \rho[G_i(N_i(x_i^n, y_i^n, z_i^n), \eta_i(u_i^n, u_i^{n+1}))\\&- G_i(N_i(x_i^{n-1}, y_i^{n-1}, z_i^{n-1}), \eta_i(u_i^n, u_i^{n+1}))]\\&+ \rho[B_i(p_i^{n-1}, M_i(u_i^{n+1})) + B_i(p_i^n, M_i(u_i^n))\\&- B_i(p_i^{n-1}, M_i(u_i^n)) - B_i(p_i^n, M_i(u_i^{n+1}))]\\\leq\ &\langle g_i(u_i^{n-1}) - g_i(u_i^n) - (K_i(u_i^{n-1}) - K_i(u_i^n)),\\&\eta_i(u_i^n, u_i^{n+1})\rangle_i\\&+ (1-\rho)\langle K_i(u_i^{n-1}) - K_i(u_i^n), \eta_i(u_i^n, u_i^{n+1})\rangle_i\\&+ \rho[G_i(N_i(x_i^{n-1}, y_i^{n-1}, z_i^{n-1}), \eta_i(u_i^{n+1}, u_i^n))\\&- G_i(N_i(x_i^n, y_i^n, z_i^n), \eta_i(u_i^{n+1}, u_i^n))]\\&+ \rho[B_i(p_i^{n-1}, M_i(u_i^{n+1})) + B_i(p_i^n, M_i(u_i^n))\\&- B_i(p_i^{n-1}, M_i(u_i^n)) - B_i(p_i^n, M_i(u_i^{n+1}))].\end{aligned} \tag{21}$$

It follows from ($\mathbf{B}_1$) and ($\mathbf{B}_3$) that

$$\begin{aligned}&\langle g_i(u_i^n) - g_i(u_i^{n+1}), \eta_i(u_i^n, u_i^{n+1})\rangle_i\\&\geq \tau_i\|u_i^{n+1} - u_i^n\|_i^2,\end{aligned} \tag{22}$$

$$\begin{aligned}&\|g_i(u_i^{n-1}) - g_i(u_i^n) - [K_i(u_i^{n-1}) - K_i(u_i^n)]\|_i\\&\leq \gamma_i\|u_i^n - u_i^{n-1}\|_i,\end{aligned} \tag{23}$$

$$\|K_i(u_i^{n-1}) - K_i(u_i^n)\|_i \leq \lambda_i\|u_i^n - u_i^{n-1}\|_i, \tag{24}$$

and

$$\|\eta_i(u_i^{n+1}, u_i^n)\|_i \leq \xi_i\|u_i^{n+1} - u_i^n\|_i. \tag{25}$$

By using ($\mathbf{B}_3$), ($\mathbf{B}_6$), ($\mathbf{C}_1$) and ($\mathbf{C}_2$), the following inequality holds:

$$\begin{aligned}&|G_i(N_i(x_i^{n-1}, y_i^{n-1}, z_i^{n-1}), \eta_i(u_i^{n+1}, u_i^n))\\&- G_i(N_i(x_i^n, y_i^n, z_i^n), \eta_i(u_i^{n+1}, u_i^n))|\\&\leq (1 + \tfrac{1}{n})\kappa_i\xi_i\|u_i^{n+1} - u_i^n\|_i\\&\cdot(\mu_i r_i\|u_1^n - u_1^{n-1}\|_1 + \theta_i s_i\|u_2^n - u_2^{n-1}\|_2\\&+ \vartheta_i t_i\|u_3^n - u_3^{n-1}\|_3).\end{aligned} \tag{26}$$

The conditions ($\mathbf{A}_2$), ($\mathbf{B}_2$) and ($\mathbf{C}_2$) imply that

$$\begin{aligned}&|B_i(p_i^{n-1}, M_i(u_i^{n+1})) + B_i(p_i^n, M_i(u_i^n))\\&- B_i(p_i^{n-1}, M_i(u_i^n)) - B_i(p_i^n, M_i(u_i^{n+1}))|\\\leq\ &(1 + \tfrac{1}{n})b_i m_i H_i(F_i(u_i^n), F_i(u_i^{n-1}))\\&\cdot\|u_i^{n+1} - u_i^n\|_i\\\leq\ &(1 + \tfrac{1}{n})b_i m_i f_i\|u_i^{n+1} - u_i^n\|_i \cdot \|u_i^n - u_i^{n-1}\|_i.\end{aligned} \tag{27}$$

It follows from (21)-(27) that

$$\begin{aligned}&\|u_i^{n+1} - u_i^n\|_i\\\leq\ &\tau_i^{-1}[\gamma_i\xi_i + |1-\rho|\lambda_i\xi_i\\&+ (1 + \tfrac{1}{n})\rho b_i f_i m_i]\|u_i^n - u_i^{n-1}\|_i\\&+ \tau_i^{-1}(1 + \tfrac{1}{n})\rho\kappa_i\xi_i \cdot (\mu_i r_i\|u_1^n - u_1^{n-1}\|_1\\&+ \theta_i s_i\|u_2^n - u_2^{n-1}\|_2 + \vartheta_i t_i\|u_3^n - u_3^{n-1}\|_3),\end{aligned}$$

and

$$\begin{aligned}&\|u_1^{n+1} - u_1^n\|_1 + \|u_2^{n+1} - u_2^n\|_2 + \|u_3^{n+1} - u_3^n\|_3\\\leq\ &\tau_1^{-1}[\gamma_1\xi_1 + |1-\rho|\lambda_1\xi_1\\&+ (1 + \tfrac{1}{n})\rho b_1 f_1 m_1]\|u_1^n - u_1^{n-1}\|_1\\&+ \rho\tau_1^{-1}(1 + \tfrac{1}{n})\kappa_1\xi_1(\mu_1 r_1\|u_1^n - u_1^{n-1}\|_1\\&+ \theta_1 s_1\|u_2^n - u_2^{n-1}\|_2 + \vartheta_1 t_1\|u_3^n - u_3^{n-1}\|_3)\\&+ \tau_2^{-1}[\gamma_2\xi_2 + |1-\rho|\lambda_2\xi_2\\&+ (1 + \tfrac{1}{n})\rho b_2 f_2 m_2]\|u_2^n - u_2^{n-1}\|_2\\&+ \rho\tau_2^{-1}(1 + \tfrac{1}{n})\kappa_2\xi_2(\mu_2 r_2\|u_1^n - u_1^{n-1}\|_1\\&+ \theta_2 s_2\|u_2^n - u_2^{n-1}\|_2 + \vartheta_2 t_2\|u_3^n - u_3^{n-1}\|_3)\\&+ \tau_3^{-1}[\gamma_3\xi_3 + |1-\rho|\lambda_3\xi_3\\&+ (1 + \tfrac{1}{n})\rho b_3 f_3 m_3]\|u_3^n - u_3^{n-1}\|_3\\&+ \rho\tau_3^{-1}(1 + \tfrac{1}{n})\kappa_3\xi_3(\mu_3 r_3\|u_1^n - u_1^{n-1}\|_1\\&+ \theta_3 s_3\|u_2^n - u_2^{n-1}\|_2 + \vartheta_3 t_3\|u_3^n - u_3^{n-1}\|_3)\\=\ &\sigma_1(n)\|u_1^n - u_1^{n-1}\|_1 + \sigma_2(n)\|u_2^n - u_2^{n-1}\|_2\\&+ \sigma_3(n)\|u_3^n - u_3^{n-1}\|_3,\end{aligned} \tag{28}$$

where for each $i \in \mathbb{J}$,

$$\sigma_i(n) = \tau_i^{-1}\xi_i(\gamma_i + |1-\rho|\lambda_i) + \tau_i^{-1}(1 + \tfrac{1}{n})\rho\alpha_i,$$

and α_i is defined by (16). It's easy to see that

$$\begin{aligned}&\|(u_1^{n+1}, u_2^{n+1}, u_3^{n+1}) - (u_1^n, u_2^n, u_3^n)\|^*\\&\leq \sigma(n)\|(u_1^n, u_2^n, u_3^n) - (u_1^{n-1}, u_2^{n-1}, u_3^{n-1})\|^*,\end{aligned} \tag{29}$$

by (28), where

$$\sigma(n) = \max\{\sigma_1(n),\ \sigma_2(n),\ \sigma_3(n)\} \to \sigma$$

as $n \to \infty$. It follows from (15) that there exist a positive number $\sigma_0 < 1$ and a sufficiently large integer $n_0 \geq 0$ such that $\sigma(n) \leq \sigma_0$ for all $n > n_0$, which implies that $\{(u_1^n, u_2^n, u_3^n)\}$ is a Cauchy sequence in $E_1 \times E_2 \times E_3$ by (29). Since $(E_1 \times E_2 \times E_3,\ \|\cdot\|^*)$ is a Banach space, $\{(u_1^n, u_2^n, u_3^n)\}$ strongly converges to some $(\bar{u}_1, \bar{u}_2, \bar{u}_3) \in E_1 \times E_2 \times E_3$. We claim that $\{x_i^n\},\ \{y_i^n\},\ \{z_i^n\}$ and $\{p_i^n\}$ are also Cauchy sequences and strongly converge $\bar{x}_i \in E_1^*,\ \bar{y}_i \in E_2^*,\ \bar{z}_i \in E_3^*$ and $\bar{p}_i \in E_i$, respectively. In fact, for each $(x_i^n, y_i^n, z_i^n) \in R_i(u_1^n) \times S_i(u_2^n) \times T_i(u_3^n)$ and $p_i^n \in F_i(u_i^n)$,

$$\begin{aligned}
&\|x_i^n - x_i^{n+1}\|_1 \leq (1 + \tfrac{1}{n+1}) r_i \|u_1^n - u_1^{n+1}\|_1,\\
&\|y_i^n - y_i^{n+1}\|_2 \leq (1 + \tfrac{1}{n+1}) s_i \|u_2^n - u_2^{n+1}\|_2,\\
&\|z_i^n - z_i^{n+1}\|_3 \leq (1 + \tfrac{1}{n+1}) t_i \|u_3^n - u_3^{n+1}\|_3,\\
&\|p_i^n - p_i^{n+1}\|_i \leq (1 + \tfrac{1}{n+1}) f_i \|u_i^n - u_i^{n+1}\|_i,
\end{aligned}$$

by (13) and ($\mathbf{C}_2$). We also see that for each $i \in \mathbb{J}$, $\bar{x}_i \in R_i(\bar{u}_1)$, since

$$\begin{aligned}
&d_1(\bar{x}_i, R_i(\bar{u}_1))\\
&\leq \|\bar{x}_i - x_i^n\|_1 + d_1(x_i^n, R_i(u_1^n))\\
&\quad + H_1(R_i(u_1^n), R_i(\bar{u}_1))\\
&\leq \|\bar{x}_i - x_i^n\|_1 + r_i\|\bar{u}_1 - u_1^n\|_1 \to 0 \ \text{ as } n \to \infty.
\end{aligned}$$

Similarly, $\bar{y}_i \in S_i(\bar{u}_2),\ \bar{z}_i \in T_i(\bar{u}_3)$ and $\bar{p}_i \in F_i(\bar{u}_i)$. Hence, $Q_n \to Q$ as $n \to \infty$, where

$$\begin{aligned}
Q = (&\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{z}_1,\\
&\bar{z}_2, \bar{z}_3, \bar{p}_1, \bar{p}_2, \bar{p}_3).
\end{aligned}$$

Rewrite (14) as follows: for all $(w_1,\ w_2,\ w_3) \in E_1 \times E_2 \times E_3$ and for each $i \in \mathbb{J}$,

$$\begin{aligned}
&\langle g_i(u_i^{n+1}) - g_i(u_i^n), \eta_i(w_i, u_i^{n+1})\rangle_i\\
&+ \rho G_i(N_i(x_i^n, y_i^n, z_i^n), \eta_i(w_i, u_i^{n+1}))\\
&+ \rho\langle K_i(u_i^n), \eta_i(w_i, u_i^{n+1})\rangle_i\\
&+ \rho B_i(z_i^n, M_i(w_i)) - \rho B_i(z_i^n, M_i(u_i^{n+1}))\\
&\geq 0.
\end{aligned} \tag{30}$$

Since $Q_n \to Q$ as $n \to \infty$ and g_i is β_i-Lipschitz continuous (See Remark 11),

$$\begin{aligned}
&|\langle g_i(u_i^{n+1}) - g_i(u_i^n), \eta_i(w_i, u_i^{n+1})\rangle_i|\\
&\leq \xi_i\beta_i(\|u_i^{n+1} - \bar{u}_i\|_i + \|u_i^n - \bar{u}_i\|_i) \cdot \|w_i - u_i^{n+1}\|_i\\
&\to 0 \text{ as } n \to \infty.
\end{aligned} \tag{31}$$

Also,

$$\begin{aligned}
&|G_i(N_i(x_i^n, y_i^n, z_i^n), \eta_i(w_i, u_i^{n+1}))\\
&\quad - G_i(N_i(\bar{x}_i, \bar{y}_i, \bar{z}_i), \eta_i(w_i, \bar{u}_i))|\\
&\leq |G_i(N_i(x_i^n, y_i^n, z_i^n), \eta_i(w_i, u_i^{n+1}))\\
&\quad - G_i(N_i(\bar{x}_i, \bar{y}_i, \bar{z}_i), \eta_i(w_i, u_i^{n+1}))|\\
&\quad + |G_i(N_i(\bar{x}_i, \bar{y}_i, \bar{z}_i), \eta_i(w_i, u_i^{n+1}))\\
&\quad - G_i(N_i(\bar{x}_i, \bar{y}_i, \bar{z}_i), \eta_i(w_i, \bar{u}_i))|\\
&\leq \kappa_i\xi_i\|w_i - u_i^{n+1}\|_i\\
&\quad \cdot (r_i\|x_i^n - \bar{x}_i\|_1 + s_i\|y_i^n - \bar{y}_i\|_2 + t_i\|z_i^n - \bar{z}_i\|_3)\\
&\quad + \kappa_i\xi_i\|N_i(\bar{x}_i, \bar{y}_i, \bar{z}_i)\|_i \cdot \|u_i^n - u_i^{n+1}\|_i\\
&\to 0 \text{ as } n \to \infty,
\end{aligned} \tag{32}$$

$$\begin{aligned}
&|\langle K_i(u_i^n), \eta_i(w_i, u_i^{n+1})\rangle_i - \langle K_i(\bar{u}_i), \eta_i(w_i, \bar{u}_i)\rangle_i|\\
&\leq |\langle K_i(u_i^n) - K_i(\bar{u}_i), \eta_i(w_i, u_i^{n+1})\rangle_i|\\
&\quad + |\langle K_i(\bar{u}_i), \eta_i(w_i, u_i^{n+1}) - \eta_i(w_i, \bar{u}_i)\rangle_i|\\
&\leq \lambda_i\xi_i\|w_i - u_i^{n+1}\|_i \cdot \|u_i^n - \bar{u}_i\|_i\\
&\quad + \xi_i\|K_i(\bar{u}_i)\|_i \cdot \|u_i^{n+1} - \bar{u}_i\|_i\\
&\to 0 \text{ as } n \to \infty,
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
&|B_i(p_i^n, M_i(w_i)) - B_i(p_i^n, M_i(u_i^{n+1}))\\
&\quad - [B_i(\bar{p}_i, M_i(w_i)) - B_i(\bar{p}_i, M_i(\bar{u}_i))]|\\
&\leq |B_i(p_i^n, M_i(w_i)) + B_i(p_i^n, M_i(\bar{u}_i))\\
&\quad - B_i(\bar{p}_i, M_i(w_i)) - B_i(\bar{p}_i, M_i(\bar{u}_i))|\\
&\quad + |B_i(p_i^n, M_i(u_i^{n+1})) - B_i(p_i^n, M_i(\bar{u}_i))|\\
&\leq b_i\|M_i(w_i) - M_i(\bar{u}_i)\|_i \cdot \|p_i^n - \bar{p}_i\|_i\\
&\quad + b_i m_i\|p_i^n\|_i \cdot \|u_i^{n+1} - \bar{u}_i\|_i\\
&\to 0 \text{ as } n \to \infty.
\end{aligned} \tag{34}$$

By letting $n \to \infty$ in (29) and applying (30)-(34), for all $(w_1, w_2, w_3) \in E_1 \times E_2 \times E_3$,

$$\begin{aligned}
&G_i(N_i(\bar{x}_i, \bar{y}_i, \bar{z}_i), \eta_i(w_i, \bar{u}_i)) + \langle K_i(\bar{u}_i), \eta_i(w_i, \bar{u}_i)\rangle_i\\
&+ B_i(\bar{p}_i, M_i(w_i)) - B_i(\bar{p}_i, M_i(\bar{u}_i)) \geq 0,\ i \in \mathbb{J},
\end{aligned}$$

that is,

$$\begin{aligned}
Q = (&\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{y}_1, \bar{y}_2, \bar{y}_3,\\
&\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{p}_1, \bar{p}_2, \bar{p}_3)
\end{aligned}$$

is a solution of the S-GMELP. □

Remark 13 *Under some suitable assumptions, the condition (15) is well-defined. For example, in the case that for each $i \in \mathbb{J}$, $\lambda_i - \alpha_i > \Theta_i$, where $\Theta_i = \xi_i(\gamma_i + \lambda_i) - \tau_i \geq 0$ by Remark 11, we can take*

$$\rho \in (\max\{\tfrac{\Theta_1}{\lambda_1 - \alpha_1}, \tfrac{\Theta_2}{\lambda_2 - \alpha_2}, \tfrac{\Theta_3}{\lambda_3 - \alpha_3}\}, 1)$$

such that (15) holds.

6 Example

The following example is given to exemplify Theorem 12

Example Suppose that $E_1 = \mathbb{R}^1 = \mathbb{R},\ E_2 = \mathbb{R}^2$ and $E_3 = \mathbb{R}^3$. For each $i \in \mathbb{J}$, let $R_i,\ F_1 : \mathbb{R}^1 \to CB(\mathbb{R}^1),\ S_i,\ F_2 : \mathbb{R}^2 \to CB(\mathbb{R}^2)$ and $T_i,\ F_3 : \mathbb{R}^3 \to CB(\mathbb{R}^3)$ be set-valued mappings, let $N_i : \mathbb{R}^1 \times \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^i,\ \eta_i : \mathbb{R}^i \times \mathbb{R}^i \to \mathbb{R}^i$ and $K_i,\ g_i,\ M_i : \mathbb{R}^i \to \mathbb{R}^i$ be single-valued mappings, and let $G_i,\ B_i : \mathbb{R}^i \times \mathbb{R}^i \to \mathbb{R}$ be bi-functions. For any

$$u_1,\ v_1,\ x_1,\ x_2,\ x_3,\ p_1,\ q_1,\ c_1,\ d_1 \in \mathbb{R}^1,$$
$$u_2,\ v_2,\ y_1,\ y_2,\ y_3,\ p_2,\ q_2,\ c_2,\ d_2 \in \mathbb{R}^2,$$

and

$$u_3,\ v_3,\ z_1,\ z_2,\ z_3,\ p_3,\ q_3,\ c_3,\ d_3 \in \mathbb{R}^3,$$

and for each $i \in \mathbb{J}$, define $R_i,\ S_i,\ T_i,\ F_i,\ N_i,\ \eta_i,\ K_i,\ g_i,\ G_i$ and B_i as follows:

$$\begin{aligned}
&R_1(u_1) = \{a : |a| \le |u_1|\},\\
&S_1(u_2) = \{a : \|a - u_2\|_2 \le 1\},\\
&T_1(u_3) = \{a : \|a\|_3 \le \|u_3\|_3\},\\
&F_1(u_1) = [\tfrac{1}{8}(u_1 - 1), \tfrac{1}{8}u_1],\\
&N_1(x_1, y_1, z_1) = \sqrt{1 + x_1^2} - y_{11} - z_{11},\\
&\eta_1(v_1, u_1) = v_1 - u_1,\\
&K_1(u_1) = u_1 + \tfrac{1}{5}\sin u_1,\\
&g_1(u_1) = M_1(u_1) = u_1,\\
&G_1(c_1, d_1) = \tfrac{1}{10}(e^{-|c_1|} - 1)d_1,\\
&B_1(p_1, q_1) = |\sin p_1|(\sqrt{1 + q_1^2} - 1);
\end{aligned}$$

$$\begin{aligned}
&R_2(u_1) = [u_1 - 9, u_1 - 8],\\
&S_2(u_2) = \{a : \|a\|_2 \le \|u_2\|_2\},\\
&T_2(u_3) = \{a : \|a - u_3\|_3 \le 1\},\\
&F_2(u_2) = \{(1, a) : 0 \le a \le \tfrac{1}{10}|u_{21}|\},\\
&N_2(x_2, y_2, z_2) = (x_2 + y_{22} + z_{21}, 0),\\
&\eta_2(v_2, u_2) = v_2 - u_2,\\
&K_2(u_2) = \tfrac{4}{5}u_2,\\
&g_2(u_2) = M_2(u_2) = u_2,\\
&G_2(c_2, d_2) = \tfrac{1}{10}(c_{21}d_{21} + c_{22}d_{22}),\\
&B_2(p_2, q_2) = p_{22}q_{22};
\end{aligned}$$

$$\begin{aligned}
&R_3(u_1) = [u_1, u_1 + 1],\\
&S_3(u_2) = \{(-1, a) : |a| \le |u_{22}|\},\\
&T_3(u_3) = [u_{31}, u_{31} + 1] \times \{(0, 0)\},\\
&F_3(u_3) = \{a : \|a - \tfrac{1}{10}u_3\|_3 \le 1\},\\
&N_3(x_3, y_3, z_3) = (0, 0, x_3 - y_{31} - z_{33}),\\
&\eta_3(v_3, u_3) = v_3 - u_3,\\
&K_3(u_3) = \tfrac{6}{5}u_3,\\
&g_3(u_3) = M_3(u_3) = u_3,\\
&G_3(c_3, d_3) = \tfrac{1}{5}c_{31}d_{31},\\
&B_3(p_3, q_3) = p_{33}q_{33},
\end{aligned}$$

where $\|\cdot\|_2$ and $\|\cdot\|_3$ are the Euclidean norms in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, and the form $*_{ij}$ is denoted by the j-th component of $*_i$. Then, we have for each $i \in \mathbb{J}$,

(i) $R_i,\ S_i,\ T_i$ and F_i satisfy $(\mathbf{C}_2)$ with $r_i = s_i = t_i = 1,\ f_1 = \frac{1}{8},\ f_2 = \frac{1}{10}$ and $f_3 = \frac{1}{10}$;

(ii) N_i satisfies $(\mathbf{C}_1)$ with $(\mu_i, \theta_i, \vartheta_i) = (1, 1, 1)$;

(iii) η_i satisfies $(\mathbf{B}_3)$-$(\mathbf{B}_5)$ with $\xi_i = 1$;

(iv) K_i and g_i satisfy $(\mathbf{B}_1)$ with $\lambda_1 = \frac{6}{5},\ \lambda_2 = \frac{4}{5},\ \lambda_3 = \frac{6}{5},\ \gamma_i = \frac{1}{5}$ and $\tau_i = 1$;

(v) M_i satisfies $(\mathbf{B}_2)$ with $m_i = 1$;

(vi) G_i satisfies $(\mathbf{B}_6)$ with $\kappa_1 = \frac{1}{10},\ \kappa_2 = \frac{1}{10}$ and $\kappa_3 = \frac{1}{5}$;

(vii) B_i satisfies $(\mathbf{A}_1)$-$(\mathbf{A}_3)$ with $b_i = 1$.

By simply calculating, it follows that

$$\begin{cases}
\sigma_1 = \frac{1}{5} + \frac{6}{5}|1 - \rho| + \frac{21}{40}\rho,\\
\sigma_2 = \frac{1}{5} + \frac{4}{5}|1 - \rho| + \frac{1}{2}\rho,\\
\sigma_3 = \frac{1}{5} + \frac{6}{5}|1 - \rho| + \frac{1}{2}\rho,
\end{cases}$$

and for all $\rho \in (\frac{16}{27}, \frac{80}{69})$, $\sigma < 1$, that is, (15) holds. It follows from theorem 12 that this S-GMELP at least has a solution. In fact,

$$(u_1, u_2, u_3, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3, p_1, p_2, p_3)$$

is a solution of S-GMELP, where

$$\begin{aligned}
&u_1 = 0 \in \mathbb{R}^1, && u_2 = (1, 0) \in \mathbb{R}^2,\\
&u_3 = (0, 0, -\tfrac{1}{6}) \in \mathbb{R}^3, &&\\
&x_1 = 0 \in R_1(u_1), && x_2 = -8 \in R_2(u_1),\\
&x_3 = 1 \in R_3(u_1), &&\\
&y_1 = (1, 0) \in S_1(u_2), && y_2 = (0, 0) \in S_2(u_2),\\
&y_3 = (-1, 0) \in S_3(u_2), &&\\
&z_1 = (0, 0, 0) \in T_1(u_3), && z_2 = (0, \tfrac{1}{2}, -\tfrac{1}{3}) \in T_2(u_3),\\
&z_3 = (\tfrac{2}{3}, 0, 0) \in T_3(u_3), &&\\
&p_1 = 0 \in F_1(u_1), && p_2 = (1, 0) \in F_2(u_2),\\
&p_3 = (0, 0, \tfrac{1}{5}) \in F_3(u_3). &&
\end{aligned}$$

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